Chapter 2

Second Order Ordinary Differential Equations

2. Linear Second Order Differential Equations with constant coefficients.

The General formula of a linear D.Es is:

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots\dots + P_n \ y = f(x)$$

where P_1 , P_2 ……. P_n , *f(x) are functions of (x)*

The General formula of a linear D.E with constant coefficients is:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots\dots + a_n \ y = f(x)$$

If f(x) = 0 then the equation is **Homogeneous**.
If f(x) ≠ 0 then the equation is **Nonhomogeneous**.

Examples:

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 5y = x \quad \Rightarrow \quad \text{Nonhomogene}$$

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 5y = 0 \quad \Rightarrow \quad \textit{Homogeneous}$$

Using Operator:

$$\frac{d}{dx} = Operater = D \ \rightarrow \frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$(D^2 + 5D + 6)y = 0$$

if y=sin(x)

$$if \ y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

2.1 Solution of Homogeneous Linear D.Es with constant coefficients:

Example:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \quad \Longrightarrow (D^2 + a_1D + a_2)y = 0$$

General solution for second-order homogeneous linear D.Es:

Case 1: ***If m_1 and m_2 are real and distinct $m_1 \neq m_2$***

$$(D^2 + a_1 D + a_2)\,y = 0$$

$$(D - m_1)(D - m_2)\,y = 0$$

$$let \quad u = (D - m_2)\,y \quad \Rightarrow \quad (D - m_1)u = 0$$

$$Du = m_1 u \quad \Rightarrow \frac{du}{dx} = m_1 u \quad \Rightarrow \quad \int \frac{du}{u} = \int m_1\, dx + c$$

$$\ln u = m_1 x + c \quad \Rightarrow \quad u = e^{m_1 x + c} = c_1 e^{m_1 x}$$

$$u = (D - m_2)\,y \quad \Rightarrow \quad (D - m_2)\,y = c_1 e^{m_1 x}$$

$$\frac{dy}{dx} - m_2\, y = c_1 e^{m_1 x} \quad \Leftrightarrow \quad linear\ D.E$$

$$\frac{dy}{dx} + P\,y = Q \quad \Rightarrow \quad P = -m_2 \qquad Q = c_1 e^{m_1 x}$$

$$I(x) = \int e^{p(x)\,dx} = \int e^{-m_2\, dx} = e^{-m_2 x}$$

$$I(x)\,y = \int I(x).Q.dx + c_2$$

$$e^{-m_2 x}.y = \int e^{-m_2 x}.c_1 e^{m_1 x}.dx + c_2$$

$$e^{-m_2 x}.y = \int c_1 e^{(m_1 - m_2).x}.dx + c_2$$

$$e^{-m_2 x}.y = \frac{1}{m_1 - m_2}.\int c_1 e^{(m_1 - m_2).x}.dx + c_2$$

$$e^{-m_2 x}.y = \frac{c_1 e^{(m_1 - m_2).x}}{m_1 - m_2} + c_2$$

$$y = \frac{c_1 e^{(m_1 - m_2).x}}{m_1 - m_2}.e^{m_2 x} + c_2.e^{m_2 x} \quad \Rightarrow \quad y = \frac{c_1 e^{m_1 x - m_2 x + m_2 x}}{m_1 - m_2} + c_2.e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} \quad where \quad m_1 \neq m_2$$

Case 2: ***If $m_1 = m_2$,***

$$(D-m)^2\,y = 0$$
$$(D-m)(D-m)\,y = 0$$
$$Let\ z = (D-m)\,y \quad \Rightarrow \quad (D-m)\,z = 0$$
$$\frac{dz}{dx} - m\,z = 0 \quad \Rightarrow \quad \frac{dz}{dx} = m\,z \quad \Rightarrow \quad \frac{dz}{z} = m\,dx \quad \Rightarrow \quad \ln z = m\,x + c_1$$
$$z = e^{m\,x}.e^{c_1} = C_1\,e^{m\,x}$$
$$\because \quad (D-m)\,y = z \quad \Rightarrow \quad (D-m)\,y = C_1\,e^{m\,x} \quad \Rightarrow \quad \frac{dy}{dx} - m\,y = C_1\,e^{m\,x} \quad \Leftrightarrow \quad linear$$
$$P = -m \qquad Q = C_1\,e^{m\,x}$$
$$I(x) = \int e^{p(x)\,dx} = \int e^{-m\,dx} = e^{-m\,x}$$
$$I(x)\,y = \int I(x).Q.dx + c_2$$
$$e^{-m\,x}.y = \int e^{-m\,x}.C_1\,e^{m\,x}.dx + c_2$$
$$e^{-m\,x}.y = (C_1\,x + c_2) \quad \Rightarrow$$

$$y = (C_1 x + C_2).\,e^{mx} \qquad where \quad m_1 = m_2$$

Case 3: ***If $m_1 = p + q\,i$ and $m_2 = p - q\,i$ the characteristic equation has a complex roots.***

$$Let\ the\ general\ \text{hom}ogeneous\ Linear\ D.E\ form\ be:$$
$$y'' + p_1\,y' + p_2\,y = 0$$
$$m_{1,2} = \frac{-p_1 \mp \sqrt{p_1^{\,2} - 4\,p_2}}{2}$$
$$Let\ p_1^{\,2} - 4\,p_2 = -4\,q^2$$
$$m_{1,2} = -\frac{p_1}{2} \mp i\,q$$
$$y = c_1\,e^{(-\frac{p_1}{2} - i\,q)\,x} + c_2\,e^{(-\frac{p_1}{2} + i\,q)\,x}$$
$$y = e^{-\frac{p_1}{2}x}\left[c_1\,e^{-i\,q\,x} + c_2\,e^{i\,q\,x}\right]$$
$$\sin x = \frac{e^{i\,x} - e^{-ix}}{2i} \quad and \quad \cos x = \frac{e^{i\,x} + e^{-ix}}{2i} \quad Euler's\ formula$$
$$e^{i\,x} = \cos x + i.\sin x$$
$$e^{-ix} = \cos x - i.\sin x$$

$$y = e^{-\frac{p_1}{2}x}\left[c_1(\cos qx - i.\sin qx) + c_2(\cos qx + i.\sin qx)\right]$$

$$y = e^{-\frac{p_1}{2}x}\left[(c_1 + c_2)\cos qx + (c_1 - c_2)\sin qx\right]$$

$$\therefore y = e^{-\frac{p_1}{2}x}(A\cos qx + B\sin qx)$$

$$y = e^{px}(A\cos qx + B\cos qx) \quad where \quad m_{1,2} = p + qi$$

Example1: Find the general solution of the equation: $y'' + 7y' + 12y = 0$?

Solution: $y'' + 7y' + 12y = 0 \Rightarrow m^2 + 7m + 12 = 0$

$$m_{1,2} = \frac{-7 \mp \sqrt{49 - 48}}{2} \Rightarrow m_1 = -4 \qquad m_2 = -3$$

$$y = C_1 e^{m_1 x} + c_2.e^{m_2 x} = C_1 e^{-4x} + c_2.e^{-3x}$$

Example2: Solve the equation: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$?

Solution: $m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \Rightarrow m_1 = 3 \qquad m_2 = 2$

$$y = C_1 e^{m_1 x} + c_2.e^{m_2 x} = C_1 e^{3x} + c_2.e^{2x}$$

Example3: Solve the equation: $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$?

Solution:

$$4m^2 + 4m + 1 = 0 \Rightarrow m^2 + m + \frac{1}{4} = 0 \Rightarrow (2m+1)^2 = 0 \Rightarrow m_1 = m_2 = -\frac{1}{2}$$

$$y = (C_1 x + c_2).e^{mx} \Rightarrow y = (c_1 x + c_2)\, e^{-\frac{1}{2}x}$$

Example4: Solve the equation: $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$?

Solution:

$$2m^2 + 3m + 4 = 0$$

$$m_{1,2} = \frac{-3 \mp \sqrt{9 - (4 \times 2 \times 4)}}{2 \times 2} = \frac{-3 \mp \sqrt{-23}}{4} = -\frac{3}{4} \mp \frac{\sqrt{23}}{4} i$$

$$y = e^{-\frac{3}{4}x}\left(A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right)$$

2.2 Initial Value and Boundary Value Problems

Example: Find the special solution of the equation: $4\frac{d^2y}{dx^2}+16\frac{dy}{dx}+17y=0$

If $y = 1$ *at* $x = 0$ *and* $y = 0$ *at* $x = \pi$?

Solution:

$$4m^2+16m+17=0 \quad \Rightarrow \quad m_{1,2}=\frac{-16\mp\sqrt{256-272}}{8}=\frac{-16\mp\sqrt{-16}}{8}=\frac{-16\mp 4i}{8}=-2\mp\frac{1}{2}i$$

$$y=e^{-2x}(A\cos\frac{x}{2}+B\sin\frac{x}{2}) \quad \Leftrightarrow \textit{the general solution}$$

$$at\ x=0 \quad y=1 \quad \Rightarrow \quad 1=e^{0}(A\cos 0+B\sin 0) \quad \Rightarrow A=1$$

$$at\ x=\pi \quad y=0 \quad \Rightarrow \quad 0=e^{-2\pi}(1\times 0+B\times 1) \quad \Rightarrow \quad B=0$$

$$\therefore y=e^{-2x}(\cos\frac{x}{2})$$

2.3 Solutions of Nonhomogeneous Linear D.E with constant coefficients:

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots\dots\dots\dots+a_n\ y=f(x)$$

The general solution to the linear nonhomogeneous differential equation is,

$y=y_h+y_p$ where y_p denotes the **particular solution** and y_h associates the **homogeneous solution**.

Two methods can be used for solving Second-order linear nonhomogeneous differential equations with constant coefficient.

2.3.1 The Method of Undetermined Coefficients:

The method is initiated by assuming a particular solution of the form:

$$y_p(x)=A_1y_1(x)+A_2y_2(x)+\dots\dots+A_ny_n(x)$$

where $A_1, A_2, \dots, A_n$ denote arbitrary multiplicative constants. These constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

Use the following table to find $y_p(x)$:

$f(x)$	$y_p(x)$
a	A
$a\,x^n$	$A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_o$
$a\,e^{px}$	$A\,e^{px}$
$a\cos\beta x$ $a\sin\beta x$	$A\cos\beta x + B\sin\beta x$
$a\,e^{px}\cos\beta x$ $a\,e^{px}\sin\beta x$	$A\,e^{px}(B_1\cos\beta x + B_2\sin\beta x)$
$a\,x^n e^{px}\cos\beta x$ $a\,x^n e^{px}\sin\beta x$	$A\,e^{px}(A_1 x^n + A_2 x^{n-1} + ... + A_n)(B_1\cos\beta x + B_2\sin\beta x)$

Example: Using the table to write y_p ?

1) $f(x) = 2x^3 \quad \Rightarrow \quad y_p = Ax^3 + Bx^2 + Cx + D$
2) $f(x) = 3e^{2x} \quad \Rightarrow \quad y_p = C\,e^{2x}$
3) $f(x) = \frac{1}{2}\sin 2x \quad \Rightarrow \quad y_p = A\cos 2x + B\sin 2x$
4) $f(x) = \sin(x) + \cos(x) \quad \Rightarrow \quad y_p = A\cos x + B\sin x$

Example-1: solve the following D.E? $y''-4y=8x^2$

Solution: $y=y_h+y_p$

$$to\ find\ y_h$$
$$Let\quad y''-4y=0 \quad\Rightarrow\quad homogeneous$$
$$m^2-4=0 \quad\Rightarrow\quad m_1=-2 \quad m_2=2$$
$$y_h=C_1e^{-2x}+C_2e^{2x}$$
$$to\ find\ y_p \quad\Leftrightarrow\quad since \quad f(x)=8x^2$$
$$y_p=Ax^2+Bx+C$$
$$y'_p=2Ax+B$$
$$y''_p=2A$$
$$Substitute\ in\ the\ original\ equation$$
$$2A-4Ax^2-4Bx-4C=8x^2$$
$$-4Ax^2-4Bx-(4C-2A)=8x^2$$
$$and\ by\ equalizing\ the\ constants\ of\ the\ two\ sides\ of\ the\ equation \quad\Rightarrow$$
$$-4A=8 \quad\Rightarrow\quad A=-2$$
$$-4B=0 \quad\Rightarrow\quad B=0$$
$$-4C+2A=0 \quad\Rightarrow\quad -4C+(2\times(-2))=0 \quad\Rightarrow\quad C=-1$$
$$\therefore y_p=-2x^2-1$$
$$y=y_p+y_h \quad\Rightarrow\quad y=C_1e^{-2x}+C_2e^{2x}-2x^2-1$$

Example-2: solve the following D.E? $y''-y'-6y=2e^{-2x}$

Solution: $y=y_h+y_p$

$$to\ find\ y_h$$
$$Let\quad y''-y'-6y=0 \quad\Rightarrow\quad homogeneous$$
$$m^2-4m-6=0 \quad\Rightarrow$$
$$m_{1,2}=\frac{-1\mp\sqrt{1+24}}{2} \quad\Rightarrow\quad m_1=-3 \quad m_2=2$$
$$y_h=C_1e^{-3x}+C_2e^{2x}$$
$$to\ find\ y_p$$
$$y_p=C\,e^{-2x}$$
$$y'_p=-2C\,e^{-2x}$$

$y''_p = 4Ce^{-2x}$

substitute in original equation

$4Ce^{-2x} - 2Ce^{-2x} - 6Ce^{-2x} = 2e^{-2x}$

$-4Ce^{-2x} = 2e^{-2x}$

$-4C = 2 \quad \Rightarrow \quad C = -\frac{1}{2}$

$\therefore y_p = -\frac{1}{2}e^{-2x}$

$y = y_h + y_p$

$y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{2}e^{-2x}$

Example-3: solve the following D.E? $y'' - y' - 2y = 10\cos x$

Solution: $y = y_h + y_p$

to find y_h

Let $y'' - y' - 2y = 0 \quad \Rightarrow \quad \text{homogeneous}$

$m^2 - m - 2 = 0 \quad \Rightarrow \quad (m+1)(m-2) = 0 \quad \Rightarrow \quad m_1 = -1 \quad m_2 = 2$

$y_h = C_1 e^{-x} + C_2 e^{2x}$

to find y_p

$y_p = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

substitute in the original equation

$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = 10\cos x$

$(-3A - B)\cos x + (A - 3B)\sin x = 10\cos x$

$-3A - B = 10 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$

$A - 3B = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$ *multiply by* 3 *and add from* 1

$-3A - B = 10$

$3A - 9B = 0$

$-10B = 10 \quad \Rightarrow B = -1$ *substitute in eq.* (1)

$-3A + 1 = 10 \quad \Rightarrow \quad A = -3$

$y_p = -3\cos x - \sin x$

$y = C_1 e^{-x} + C_2 e^{2x} - 3\cos x - \sin x$

Example-4: solve the following D.E? $y''-3y'+2y=e^x$

Solution: $y = y_h + y_p$

to find y_h

Let $y''-3y'+2y=0 \Rightarrow$ hom*ogeneous*

$m^2-3m+2=0 \Rightarrow (m-1)(m-2)=0 \Rightarrow m_1=1 \quad m_2=2$

$y_h = C_1e^x + C_2e^{2x}$

to find y_p

$y_p = C\,e^x \otimes$ *but there is a term in the eq. of* y_h *which is also* $C\,e^x$ *so we multiply* y_p *by* x

$\therefore \quad y_p = C\,x\,e^x$

$y'_p = C\,xe^x + Ce^x = C(e^x + xe^x)$

$y''_p = C\,xe^x + Ce^x + Ce^x = C(2e^x + xe^x)$

substitute in the original equation

$C(2+x)e^x - 3C(1+x)e^x + 2C\,x\,e^x = e^x$

$2C\,e^x + C\,x\,e^x - 3C\,e^x - 3C\,x\,e^x + 2C\,x\,e^x = e^x$

$-C\,e^x = e^x \quad \Rightarrow \quad C=-1$

$\therefore \quad y_p = -x\,e^x \quad \Rightarrow y = C_1\,e^x + C_2\,e^{2x} - x\,e^x$

2.3.2 Method of Variation of Parameters:

In general, if *f(x)* is not one of the types of functions considered in the (undetermined coefficients method), or if the differential equation ***does not have constant coefficient***, then this method is preferred.

Variation of parameters is another method for finding a particular solution of the second-order linear differential equation. It can be applied to all linear D.E's. It is therefore more powerful than the undetermined coefficients which is restricted to linear D.E's with constant coefficients and particular forms of *f(x)*.

The general form of the second-order linear D.E is

$$ay'' + by' + cy = f(x) \qquad (1)$$

The solution as we know is $y = y_h + y_p$ where y_h is the general solution of the corresponding homogeneous equation $f(x)=0$ which is expressed as:

$$ay'' + by' + cy = 0$$

and y_p is the particular solution, which can be expressed in this case as:

$$y_p = v_1 y_1 + v_2 y_2$$

The general method for finding a particular solution of the nonhomogeneous equation (1) above, once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring that the

$$v'_1 y_1 + v'_2 y_2 = 0$$

Then we have,

$$v_1' y_1' + v_2' y_2' = f(x)$$

Now, for the unknown functions v'_1 and v'_2, The usual procedure for solving this simple system is to use the *method of determinants* (also known as *Cramer Rule)*

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \times y_2' - y_2 \times y_1'$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = 0 \times y_2' - y_2 \times f(x)$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1 \times f(x) - 0 \times y_1'$$

then,

$v_1{}' = \frac{w_1}{w}$ and $v_2{}' = \frac{w_2}{w}$

after that,

$$v_1(x) = \int v_1{}'\,dx$$

$$v_2(x) = \int v_2{}'\,dx$$

finally,

$$y_p = v_1 y_1 + v_2 y_2$$

and the general solution is:

$$y = y_h + y_p$$

Example-1: Find the general solution to the equation

$$y'' + y = \tan x$$

Solution: The solution of the homogeneous equation

$$y'' + y = 0$$

$$y_h = A\cos x + B\sin x$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$ then,

$$v'_1 y_1 + v'_2 y_2 = 0 \qquad \Rightarrow \qquad v_1{}'\cos x + v_2{}'\sin x = 0$$

$$v_1'{}^{y_1{}'} + v_2'{}^{y_2{}'} = f(x) \qquad \Rightarrow \qquad -v_1{}'\sin x + v_2{}'\cos x = \tan x$$

then,

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 \times y_2' - y_2 \times y_1{}'$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos x \times \cos x + \sin x \times \sin x = \cos^2 x + \sin^2 x$$

$$w_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \times \tan x$$

$$w_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \times \tan x$$

then,

$$v_1' = \frac{w_1}{w} = \frac{-\sin x \times \tan x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}$$

$$v_2' = \frac{w_2}{w} = \frac{\cos x \times \tan x}{\cos^2 x + \sin^2 x} = \sin x$$

after that,

$$v_1(x) = \int v_1{}'\,dx = \int \frac{-\sin^2 x}{\cos x}\,dx = -\int (\sec x - \cos x)\,dx$$

$$v_1(x) = -\ln(\sec x + \tan x) + \sin x$$

$$v_2(x) = \int v_2{}'\,dx = \int \sin x\,dx = -\cos x$$

Finally,

$$y_p = v_1 y_1 + v_2 y_2 = [-\ln(\sec x + \tan x) + \sin x]\cos x + (-\cos x)\sin x$$

$$y_p = (-\cos x)\ln(\sec x + \tan x)$$

$$y = y_h + y_p = A\cos x + B\sin x - (\cos x)\ln(\sec x + \tan x)$$

Example2: $y'' + y' - 2y = xe^x$

Solution:

$$m^2 + m - 2 = 0$$

$$y_h = c_1 e^{-2x} + c_2 e^x$$

Since $y_1(x) = e^{-2x}$ and $y_2(x) = e^x$ then,

$$v'_1 y_1 + v'_2 y_2 = 0 \qquad \Rightarrow \qquad v_1{}' e^{-2x} + v_2{}' e^x = 0$$

$$v'_1 y_1{}' + v'_2 y_2{}' = f(x) \qquad \Rightarrow \qquad -2v_1{}' e^{-2x} + v_2{}' e^x = xe^x$$

then,

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 \times y'_2 - y_2 \times y_1{}'$$

$$w = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = 3e^{-x}$$

$$w_1 = \begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix} = -xe^{2x}$$

$$w_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix} = xe^{-x}$$

then,

$$v'_1 = \frac{w_1}{w} = \frac{-xe^{2x}}{3e^{-x}} = \frac{-1}{3} xe^{3x}$$

$$v'_2 = \frac{w_2}{w} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}$$

after that,

$$v_1(x) = \int v_1'\,dx = \int \frac{-1}{3} x e^{3x} dx = -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3} dx\right)$$

$$v_1(x) = \frac{1}{27}(1-3x)e^{3x}$$

$$v_2(x) = \int v_2'\,dx = \int \frac{x}{3} dx = \frac{x^2}{6}$$

Finally,

$$y_p = v_1 y_1 + v_2 y_2 = \left[\frac{(1-3x)e^{3x}}{27}\right] e^{-2x} + \left(\frac{x^2}{6}\right) e^x$$

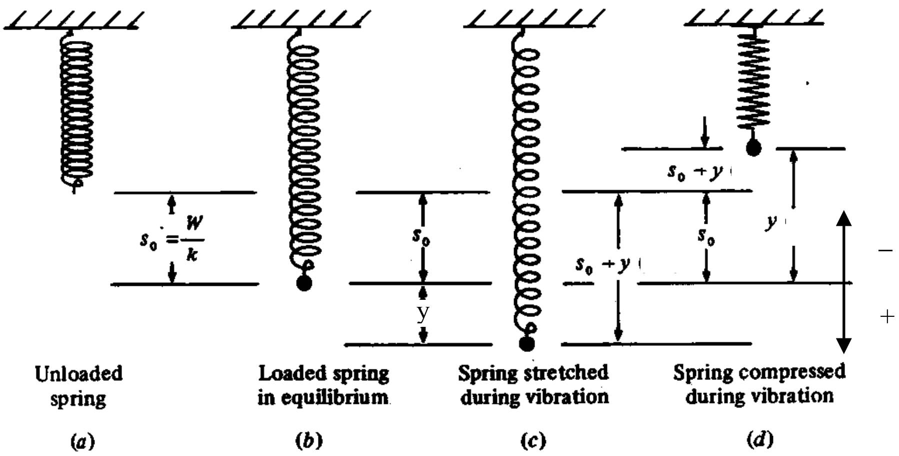
$$y_p = \frac{1}{27} e^x - \frac{1}{9} x e^x + \frac{1}{6} x^2 e^x$$

$$\therefore y = y_h + y_p = c_1 e^{-2x} + c_2 e^x - \frac{1}{9} x e^x + \frac{1}{6} x^2 e^x$$

where the term *(1/27)*e^xin y_p has been absorbed into the term $C_2 e^x$ in the complementary solution.

Applications of Second Order Linear Differential Equations with constant coefficients:

Free Oscillation:



Static Case:

$$\sum fy = 0$$

$$m\,g - k\,s_o = 0$$

Dynamic Case:

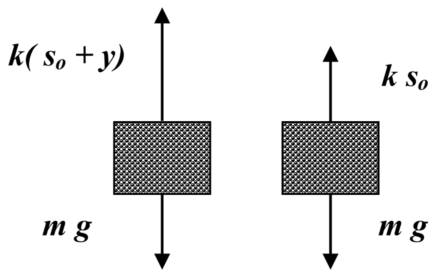
$$F = m\frac{d^2y}{dt^2}$$

$$m\,g - k\,(s_o + y) = m\,y''$$

$$but\; m\,g = k\,s_o \quad \Rightarrow$$

$$-k\,y = m\,y'' \quad \Rightarrow$$

$$\boxed{m\,y'' + k\,y = 0}$$



$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \;\Rightarrow\; \frac{d^2y}{dt^2} + \omega^2\,y = 0 \; where$$

$$\omega^2 = \frac{k}{m} \quad or \quad \omega = \sqrt{\frac{k}{m}}$$

$$m^2 + \omega^2 = 0$$

$$m^2 = -\omega^2$$

$$m = \mp\omega i$$

$$\boxed{y = A\cos\omega t + B\sin\omega t}$$

Case I (General case):

$y(0) = y_o \quad \Leftrightarrow \quad y'(0) = 0$

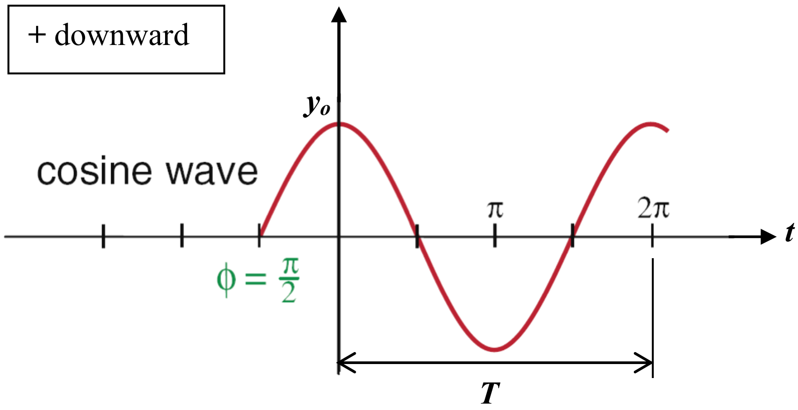
$A = y_o \quad B = 0 \quad \Rightarrow$

$$y = y_o \cos \omega t$$

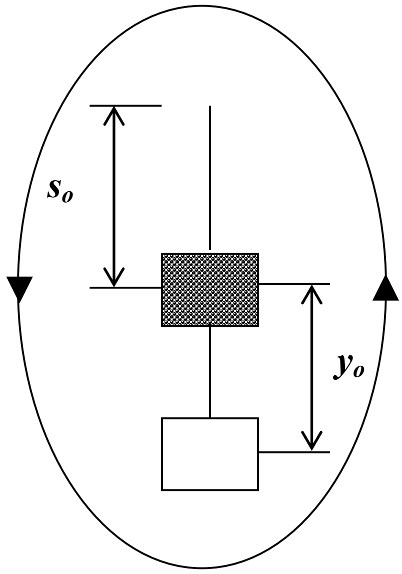
for a complete cycle

$\omega t = 2\pi \quad \Rightarrow \quad t = \dfrac{2\pi}{\omega}$

$\therefore T = \dfrac{2\pi}{\omega}$ *cycle period*



$\cos\phi = \cos(\phi + 2\pi) \quad \Rightarrow \quad \cos\omega t = \cos(\omega t + 2\pi)$



Complete Cycle

Case II:

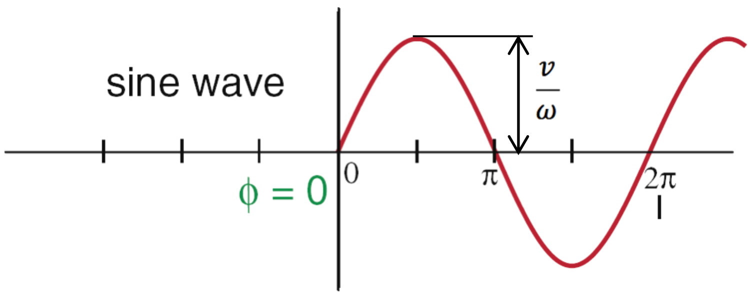
$y(0) = 0 \quad \Leftrightarrow \quad y'(0) = v$

$0 = A + 0 \quad \Rightarrow \quad A = 0$

$\dfrac{dy}{dt} = \omega B \cos \omega t$

$\left.\dfrac{dy}{dt}\right|_{t=o} = \omega B = v$

$$y = \frac{v}{\omega} \sin \omega t$$



$$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T \propto \frac{1}{\omega}$$

<u>Case III:</u>

$$y(0) = y_o \qquad y'(0) = v$$

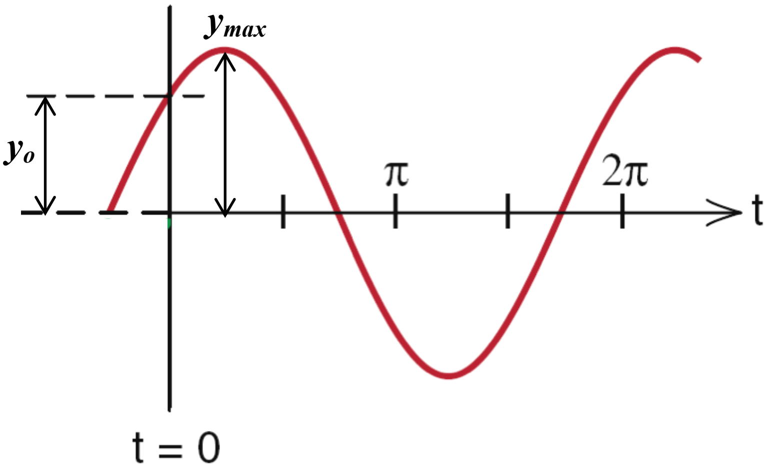
$$y_o = A + 0 \quad \Rightarrow \quad A = y_o$$

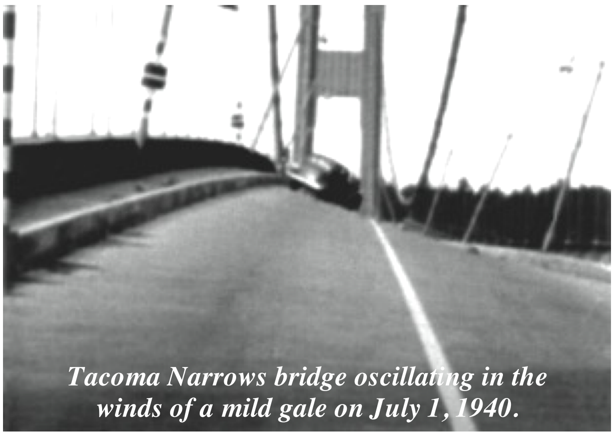
$$\frac{dy}{dt} = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$v = 0 + \omega B$$

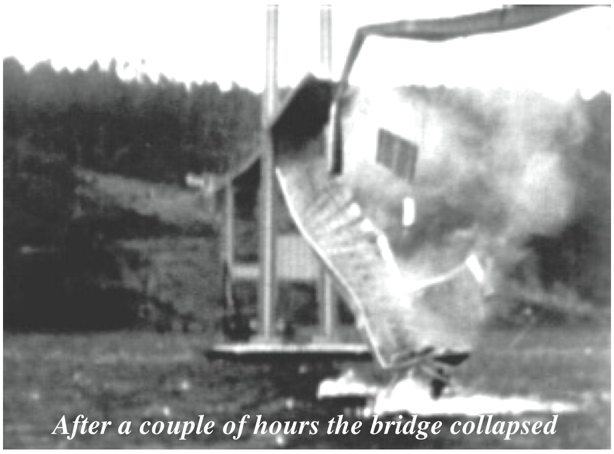
$$B = \frac{v}{w}$$

$$\boxed{y = y_o \cos \omega t + \frac{v}{\omega} \sin \omega t}$$





Tacoma Narrows bridge oscillating in the winds of a mild gale on July 1, 1940.



After a couple of hours the bridge collapsed

Example: A weight of (*7 N*) is suspended from a spring of modulus (***k=36/35 N/cm***). At ***t = 0***, while the weight in static equilibrium it is given suddenly an initial velocity of (***48 cm/sec***) in downward.

a. Find the vertical displacement as a function of ***t***.
b. What are the period and frequency of motion?
c. Through what amplitude does the weight moves?
d. At what time does the load reach its extreme displacement above and below its equilibrium position?

Solution:

a.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{36/35}{980}} = 12$$

$$y = A\cos 12t + B\sin 12t$$

$$y(0) = A + 0 \quad \Rightarrow \quad A = 0$$

$$y = B\sin 12t$$

$$y' = 12\ B\cos 12t$$

$$y'\big|_{t=o} = 12\ B \quad \Rightarrow \quad 48 = 12\ B \quad \Rightarrow \quad B = 4$$

$$\boxed{y = 4\sin 12t}$$

b.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{12} = \frac{\pi}{6} \quad \sec/cycle$$

$$f = \frac{6}{\pi} \quad cycle/\sec \quad (Hertz)$$

c.

$$y = 4\sin 12\,t \qquad -1 \le \sin 12\,t \le 1$$

$$y_{\max} = 4 \qquad y_{\min} = -4$$

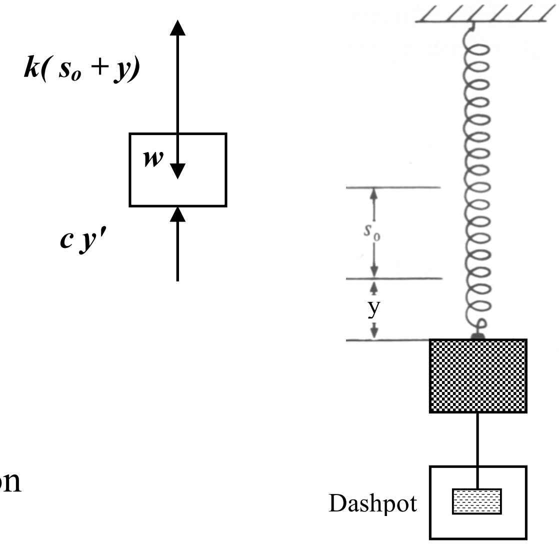
$$Amplitude = 4 + |-4| = 8$$

d.

$$\sin 12\,t = \mp 1$$

$$12t = (2n+1)\frac{\pi}{2} \quad \Leftrightarrow \quad t = \left(\frac{1+2n}{24}\right)\pi \qquad n = 0, 1, 2, \quad Multiplication\ of\ \frac{\pi}{2}$$

Damped Oscilation:



Damping $\propto \frac{dy}{dt} = c\,y'$

$$\oplus \downarrow \sum f\,y = M\,y''$$
$$w - k(s_o + y) - c\,y' = M\,y''$$
$$k\,y + M\,y'' + cy' = 0$$

$$\boxed{M\,y'' + c\,y' + k\,y = 0}$$ Arithmetic Model

$M\,m^2 + c\,m + k = 0$ Characteristic Equation

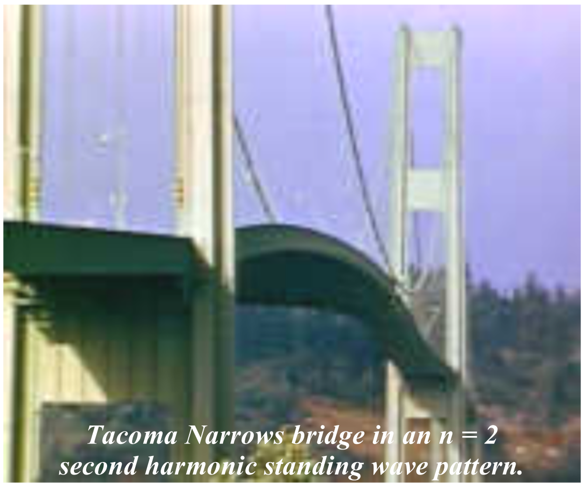
$$m_{1,2} = \frac{-c \mp \sqrt{c^2 - 4M\,k}}{2M} = \frac{-c}{2M} \mp \frac{1}{2M}\sqrt{c^2 - 4M\,k}$$
$$m_{1,2} = -\alpha \mp \beta$$
$$\alpha = \frac{c}{2M} \qquad \beta = \frac{1}{2M}\sqrt{c^2 - 4M\,k}$$

Critical Damping Coefficient:

The Critical Damping Coefficient is the value of ***c*** which makes:

$$\sqrt{c^2 - 4M\,k} = 0$$

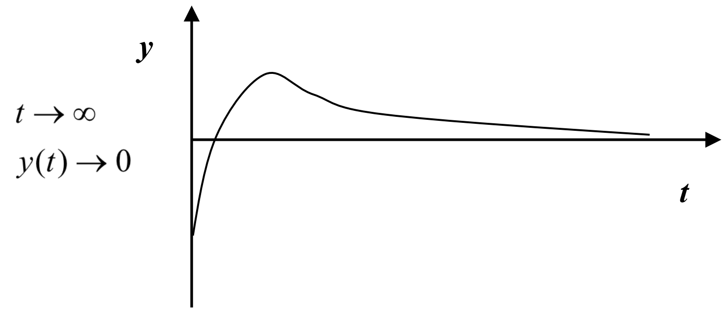
$$\boxed{C_{cr} = 2\sqrt{k\,M}}$$



Tacoma Narrows bridge in an n = 2 second harmonic standing wave pattern.

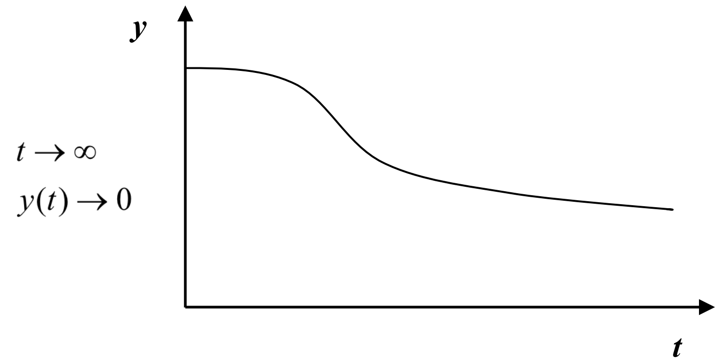
Case 1: $c \succ C_{cr}$ i.e $\sqrt{c^2 - 4Mk} \succ 0 \Rightarrow$ *Over Damping* $\Leftrightarrow$ α *and* β *are real*

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$



Case 2: $c = C_{cr}$ i.e $m_1 = m_2 = -\alpha \Rightarrow$ *Ciritical Damping*

$$y(t) = (c_1 + c_2\, t)\, e^{-\alpha t}$$



Case 3: $c \prec C_{cr} \Leftrightarrow$ *Under Damping*

$$m_{1,2} = \frac{-c}{2M} \mp \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

$$= -\alpha \mp \beta$$

$$\alpha = \frac{c}{2M} \quad \beta = \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

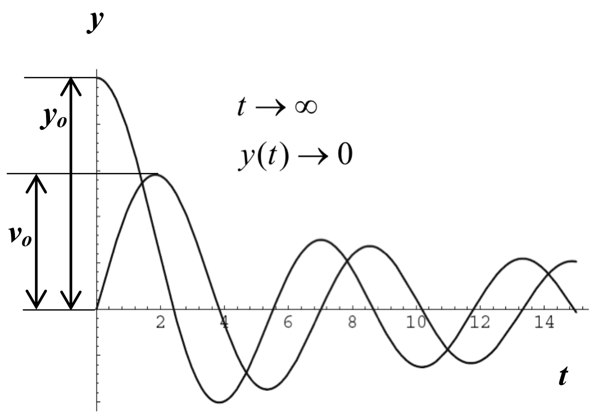
$$= \frac{1}{2M}\sqrt{4Mk - c^2} \times i$$

$$= \omega^* i$$

$$m_{1,2} = -\alpha \mp \omega^* i$$

$$y(t) = e^{pt}(A\cos qt + B\sin qt)$$

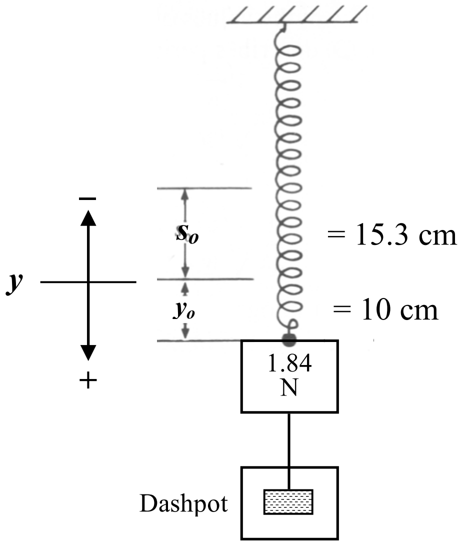
$$y = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$



Example: A ***(1.84 N)*** body is suspended by a spring which is stretched ***(15.3 cm)*** when it is loaded. If the body is drawn down ***(10 cm)*** from the position of equilibrium; find the position of the spring as a function of time ***(t)*** if:

1. c = 1.5
2. c = 3.75
3. c = 3

Solution:



$$M\,y''+c\,y'+k\,y=0$$

$$M=\frac{w}{g}=\frac{1.84}{9.8}=0.188\ kg$$

$$k=\frac{w}{s_o}=\frac{1.84}{0.153}=12\ \frac{N}{m}$$

1) for c = 1.5

$$0.188\,y''+1.5\,y'+12\,y=0$$

$$y''+8\,y'+64\,y=0$$

$$m^2+8m+64=0$$

$$m_{1,2}=\frac{-8\mp\sqrt{64-4\times 64}}{2}=-4\mp 4\sqrt{3}\,i \quad \Leftrightarrow \quad \textit{Under Damping}$$

$$y(t)=e^{-4t}(A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t)$$

$$\underline{\textit{Initial Conditions}}:$$

$$at \quad t=0 \quad \Rightarrow \quad y'=o \quad \Rightarrow y=0.1m$$

$$y(0)=A+0=0.1 \quad \Rightarrow \quad A=0.1$$

$$y'(t)=\left[-4\sqrt{3}\,A\sin 4\sqrt{3}\,t+4\sqrt{3}\,B\cos 4\sqrt{3}\,t\right]+\left[A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t\right](-4e^{-4t})$$

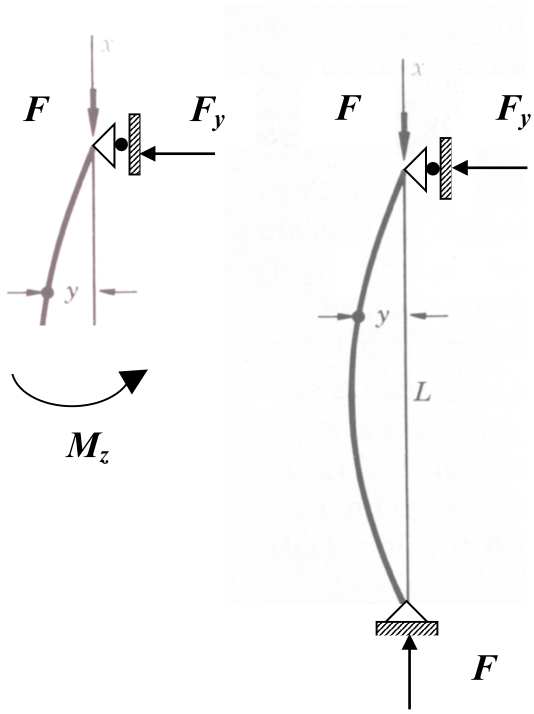
$$y'(0)=4\sqrt{3}\,B-4A=4\sqrt{3}\,B-4\times 0.1=0 \quad \Rightarrow \quad B=\frac{1}{10\sqrt{3}}$$

$$y=e^{-4t}(0.1\cos 4\sqrt{3}\,t+\frac{1}{10\sqrt{3}}\sin 4\sqrt{3}\,t)$$

2) for c = 3.75

3) for c = 3

Column Buckling:



$$\sum f_y = 0 \quad \Rightarrow \quad F_y = 0$$

$$M = -F\,y$$

$$\frac{d^2 y}{d x^2} = \frac{M}{E I}$$

$$y'' = -\frac{F}{E I} y \quad \Rightarrow \quad y'' + \frac{F}{E I} y = 0$$

$$m^2 + \frac{F}{E I} = 0 \quad \Rightarrow \quad m^2 = -\frac{F}{E I}$$

$$m_{1,2} = \mp \sqrt{\frac{F}{E I}}\, i$$

$$y = A \cos\sqrt{\frac{F}{E I}}\, x + B \sin\sqrt{\frac{F}{E I}}\, x$$

$$at \quad x = 0 \qquad y = 0$$

$$0 = A + B \times 0 \quad \Rightarrow \quad A = 0$$

$$\therefore\ y = B \sin\sqrt{\frac{F}{E I}}\, x$$

at $x = L$ $y = o$ *also*

but this means that $B = 0$ *which results in zero equation. However,*

If the load F have just the right value to make:

$\sqrt{\frac{F}{E I}}\, L = n\,\pi$ *then the last equation will be satisfied without B being* 0 *and*

the equilibrium is possible in a deflected position defined by:

$$\sqrt{\frac{F}{E I}} = \frac{n\,\pi}{L} \qquad n = 1, 2, 3, \ \ldots\ldots\ldots \quad \Rightarrow \quad y = B \sin\frac{n\,\pi\,x}{L}$$

$$\sqrt{\frac{F_n}{E I}} = \frac{n\,\pi}{L} \quad \Rightarrow \quad \frac{F_n}{E I} = \left(\frac{n\pi}{L}\right)^2 \quad \Rightarrow \quad F_n = \left(\frac{n\pi}{L}\right)^2 E I$$

$$for\ n = 1 \quad \Rightarrow \quad F_1 = \frac{\pi^2 E I}{L^2}$$

For values of F below the lowest critical load, the column will remain in its undeflected vertical position, or if displaced slightly from it, will return to it as an equilibrium configuration. For values of F above the lowest critical load and different from the higher critical loads, the column can theoretically remain in a vertical position, but the equilibrium is unstable, and if the column is deflected slightly, it will not return to a vertical position but will continue to deflect until it collapses. Thus, only the lowest critical load is of much practical significance.