Multiple Integral

Double Integral

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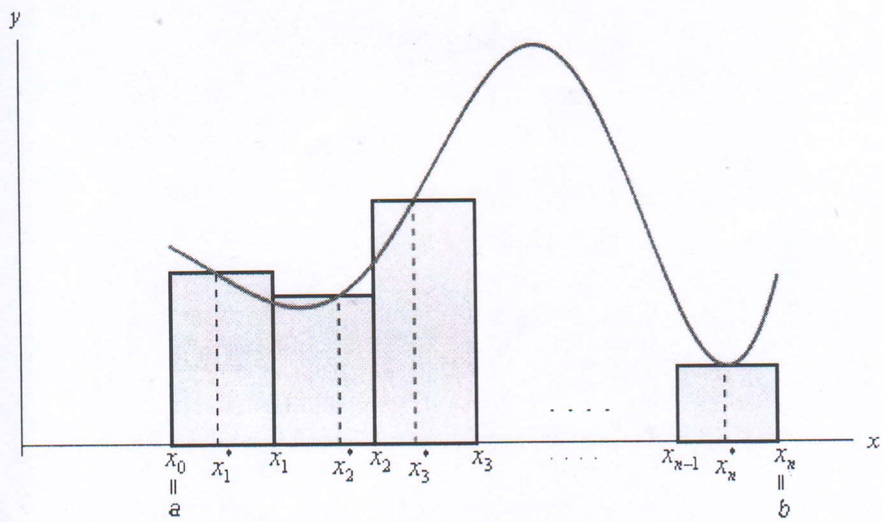
Double Integrals

Double Integrals

The definition of a definite integrals for functions of single variables

$\int_a^b f(x)\,dx$ and $b \le x \le a$. also $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

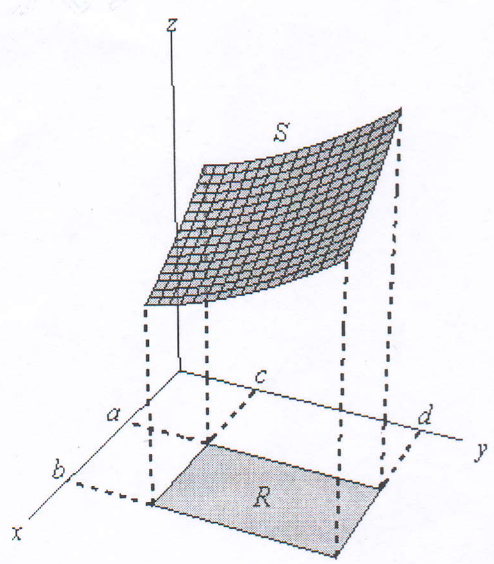


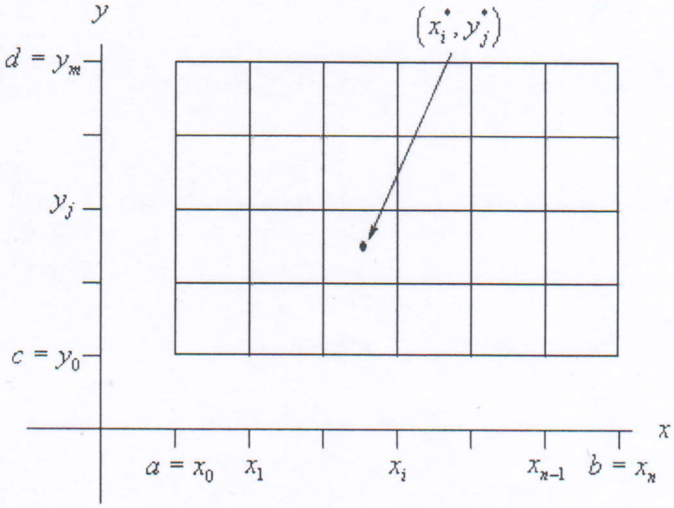
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b]\times[c,d]$$

This means that the ranges for x and y are $a \le x \le b$ and $c \le y \le d$.

Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case.





Here is the official definition of a double integral of a function of two variables over a rectangular region R as well as the notation that we'll use for it.

$$\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

Iterated Integrals

The following theorem tells us how to compute a double integral over a rectangle.

Fubini's Theorem

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$ then,

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

These integrals are called **iterated integrals**.

Example 1 Compute each of the following double integrals over the indicated rectangles.

(a) $\iint_R 6xy^2\,dA$, $R=[2,4]\times[1,2]$

(b) $\iint_R 2x-4y^3\,dA$, $R=[-5,4]\times[0,3]$

(c) $\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA$, $R=[-2,-1]\times[0,1]$

(d) $\iint_R \frac{1}{(2x+3y)^2}\,dA$, $R=[0,1]\times[1,2]$

(e) $\iint_R xe^{xy}\,dA$, $R=[-1,2]\times[0,1]$

Solution

(a) $\iint\limits_R 6xy^2\,dA\,,\quad R=[2,4]\times[1,2]$

Solution 1

In this case we will integrate with respect to y first. So, the iterated integral that we need to compute is,

$$\iint\limits_R 6xy^2\,dA=\int_2^4\int_1^2 6xy^2\,dy\,dx$$

$$\iint\limits_R 6xy^2\,dA=\int_2^4\left(2xy^3\right)\Big|_1^2\,dx$$

$$=\int_2^4 16x-2x\,dx$$

$$=\int_2^4 14x\,dx$$

$$\iint\limits_R 6xy^2\,dA=7x^2\Big|_2^4=84$$

Solution 2

In this case we'll integrate with respect to x first and then y. Here is the work for this solution.

$$\iint\limits_R 6xy^2\,dA=\int_1^2\int_2^4 6xy^2\,dx\,dy$$

$$=\int_1^2\left(3x^2y^2\right)\Big|_2^4\,dy$$

$$=\int_1^2 36y^2\,dy$$

$$=12y^3\Big|_1^2$$

$$=84$$

(b) $\iint\limits_R 2x-4y^3\,dA\,,\quad R=[-5,4]\times[0,3]$

For this integral we'll integrate with respect to y first.

$$\iint\limits_R 2x-4y^3\,dA=\int_{-5}^4\int_0^3 2x-4y^3\,dy\,dx$$

$$=\int_{-5}^4\left(2xy-y^4\right)\Big|_0^3\,dx$$

$$=\int_{-5}^4 6x-81\,dx$$

$$=\left(3x^2-81x\right)\Big|_{-5}^4$$

$$=-756$$

(c) $\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA\,,\quad R=[-2,-1]\times[0,1]$

$$\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA=\int_0^1\int_{-2}^{-1}x^2y^2+\cos(\pi x)+\sin(\pi y)\,dx\,dy$$

$$=\int_0^1\left(\frac{1}{3}x^3y^2+\frac{1}{\pi}\sin(\pi x)+x\sin(\pi y)\right)\Bigg|_{-2}^{-1}dy$$

$$=\int_0^1\frac{7}{3}y^2+\sin(\pi y)\,dy$$

$$=\frac{7}{9}y^3-\frac{1}{\pi}\cos(\pi y)\Bigg|_0^1$$

$$=\frac{7}{9}+\frac{2}{\pi}$$

(d) $\iint\limits_R \frac{1}{(2x+3y)^2}\,dA\,,\quad R=[0,1]\times[1,2]$

$$\iint\limits_R (2x+3y)^{-2}\,dA=\int_1^2\int_0^1(2x+3y)^{-2}\,dx\,dy$$

$$=\int_1^2\left(-\frac{1}{2}(2x+3y)^{-1}\right)\Bigg|_0^1dy$$

$$=-\frac{1}{2}\int_1^2\frac{1}{2+3y}-\frac{1}{3y}\,dy$$

$$=-\frac{1}{2}\left(\frac{1}{3}\ln|2+3y|-\frac{1}{3}\ln|y|\right)\Bigg|_1^2$$

$$=-\frac{1}{6}(\ln 8-\ln 2-\ln 5)$$

(e) $\iint\limits_R x\mathbf{e}^{xy}\,dA\,,\quad R=[-1,2]\times[0,1]$

$$\iint\limits_R x\mathbf{e}^{xy}\,dA=\int_{-1}^2\int_0^1x\mathbf{e}^{xy}\,dy\,dx$$

be done with the quick substitution,

$$u=xy\qquad\qquad du=x\,dy$$

$$\iint\limits_R x\mathbf{e}^{xy}\,dA=\int_{-1}^2\mathbf{e}^{xy}\Big|_0^1dx$$

$$=\int_{-1}^2\mathbf{e}^x-1\,dx$$

$$=\left(\mathbf{e}^x-x\right)\Big|_{-1}^2$$

$$=\mathbf{e}^2-2-\left(\mathbf{e}^{-1}+1\right)$$

$$=\mathbf{e}^2-\mathbf{e}^{-1}-3$$

If we change the order from dydx to dxdy the solution will be more difficult see that

$$\iint_R x\mathbf{e}^{xy}\,dA = \int_0^1\int_{-1}^2 x\mathbf{e}^{xy}\,dx\,dy$$

In order to do this we would have to use integration by parts as follows,

$$u = x \qquad dv = \mathbf{e}^{xy}\,dx$$

$$du = dx \qquad v = \frac{1}{y}\mathbf{e}^{xy}$$

The integral is then,

$$\iint_R x\mathbf{e}^{xy}\,dA = \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\,dx\right)\Bigg|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy}\right)\Bigg|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y}\right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y}\right) dy$$ ———— difficult to continue

Fact

If $f(x,y) = g(x)h(y)$ and we are integrating over the rectangle $R = [a,b]\times[c,d]$ then,

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Example 2 Evaluate $\iint_R x\cos^2(y)\,dA$, $R = [-2,3]\times\left[0,\frac{\pi}{2}\right]$.

Solution

Since the integrand is a function of x times a function of y we can use the fact.

$$\iint_R x\cos^2(y)\,dA = \left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos^2(y)\,dy\right)$$

$$= \left(\frac{1}{2}x^2\right)\Bigg|_{-2}^{3}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos(2y)\,dy\right)$$

$$= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y + \frac{1}{2}\sin(2y)\right)\Bigg|_0^{\frac{\pi}{2}}\right)$$

$$= \frac{5\pi}{8}$$