First-Order Linear Differential Equations:

A First order linear differential equation is an equation of the form

$$y' + P(x)y = Q(x).$$

Where P and Q are functions of x. If the equation is written in this form it is called standard form. The equation is called *first order* because it only involves the function y and first derivatives of y. We can solve this equation in general but it is better to understand *how* to solve it than it is to just memorize the solution.

Solving $y' + P(x)y = Q(x)$:

The general solution to the first order linear differential equation is given by

$$y(x) = \frac{1}{u(x)} \int u(x)Q(x)dx,$$

with

$$u(x) = e^{\int P(x)dx}.$$

Now let's do an example.

Example: $y' + 4xy = x$

First we note that this is already in standard form with $P(x) = 4x$, and $Q(x) = x$.
The first step is to find the integrating factor

$$u(x) = e^{\int P(x)dx} = e^{\int 4xdx} = e^{2x^2}.$$

Note that we do not need the general form of the integral until the last step.
The next step is to find $y = \frac{1}{u(x)} \int u(x)Q(x)dx$.
Recall that $\frac{1}{e^{2x^2}} = e^{-2x^2}$.

Example: $y' + 4xy = x$ continued

We have $y = \frac{1}{u(x)} \int u(x)Q(x)dx$, with $u(x) = e^{2x^2}$, so we have to solve

$$y = e^{-2x^2} \int xe^{2x^2} dx.$$

This integral can be done with the method of substitution, let $z = 2x^2$, then $dz = 4xdx$

$$\int xe^{2x^2} dx = \int \frac{1}{4} e^z dz = \frac{1}{4} e^z + C = \frac{1}{4} e^{2x^2} + C.$$

Almost done, now let

$$y = e^{-2x^2} [\frac{1}{4} e^{2x^2} + C] = \frac{1}{4} + Ce^{-2x^2}.$$

Another Example:

Find the solution to

$$xy' + y = x^2 + 1$$

First we write

$$y' + \frac{1}{x}y = x + \frac{1}{x} \Rightarrow P(x) = \frac{1}{x} \text{ and } Q(x) = x + \frac{1}{x}.$$

Therefore

$$u(x) = e^{\int P(x)dx} = x,$$

and

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{x} \int x^2 + 1dx = \frac{1}{3}x^2 + 1 + Cx^{-1}.$$

Let's check our answer $y = \frac{1}{3}x^2 + 1 + Cx^{-1}$:

$$y' = \frac{2}{3}x - Cx^{-2}$$

$$\Rightarrow xy' + y = \frac{2}{3}x^2 - Cx^{-1} + \frac{1}{3}x^2 + 1 + Cx^{-1}$$

$$\Rightarrow xy' + x = x^2 + 1!$$

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Exercises:

1 – 4 Find the general solution of each equation below.

1. $y' - t^2 y = 4t^2$

2. $y' + 10y = t^2$

3. $\frac{1}{t^2} y' - e^{t^3} y = 0$

4. $y' - y = 2e^t$

5 – 16 Solve each initial value problem. What is the largest int which a unique solution is guaranteed to exist?

5. $y' + 2y = t e^{-t}$, $\quad y(0) = 2$

6. $y' - 11y = 4e^{6t}$, $\quad y(0) = 9$

7. $t y' - y = t^2 + t$, $\quad y(1) = 5$

8. $(t^2 + 1) y' - 2ty = t^3 + t$, $\quad y(0) = -4$

9. $y' + (2t - 6t^2) y = 0$, $\quad y(0) = -8$

10. $t^2 y' + 4ty = \frac{2}{t}$, $\quad y(-2) = 0$

11. $(t^2 - 49) y' + 4ty = 4t$, $\quad y(0) = 1/7$

12. $y' - y = t^2 + t$, $\quad y(0) = 3$

13. $y' + y = e^t$, $\quad y(0) = 1$

14. $t y' + 4y = 4$, $\quad y(-2) = 6$

15. $\tan(t) y' - \sec(t) \tan^2(t) y = 0$, $\quad y(0) = \pi$

16. $(t^2 + 1) y' + 2ty = 0$, $\quad y(3) = -1$