University of Anbar
College of Engineering
Dept. of Electrical Engineering

Separable Differential Equations

A first order differential equation is *separable* if it can be written in the for

$$M(x) + N(y)\,y' = 0,$$

where $M(x)$ is a function of the independent variable x only, and $N(y)$ is a function of the dependent variable y only. It is called separable because the independent and dependent variables could be moved to separate sides of tl equation:

$$N(y)\frac{dy}{dx} = -M(x)\,.$$

Multiplying through by dx,

$$N(y)dy = -M(x)dx.$$

A general solution of the equation can then be found by simply integrating both sides with respect to each respective variable:

$$\int N(y)dy = -\int M(x)dx + C\,.$$

This is the *implicit* general solution of the equation, where y is defined implicitly as a function of x by the above equation relating the antiderivatives, with respect to their individual variables, of $M(x)$ and $N(y)$.

An *explicit* general solution, in the form of $y = f(x)$, where y is explicitly defined by a function $f(x)$ which itself satisfies the original differential equation, could be found (in theory, although not always in practice) by simplifying the implicit solution and solve for y.

Separable Differential Equations

A first order differential equation is *separable* if it can be written in the for:

$$M(x) + N(y)\,y' = 0,$$

where $M(x)$ is a function of the independent variable x only, and $N(y)$ is a function of the dependent variable y only. It is called separable because th independent and dependent variables could be moved to separate sides of tl equation:

$$N(y)\frac{dy}{dx} = -M(x)\,.$$

Multiplying through by dx,

$$N(y)dy = -M(x)dx.$$

A general solution of the equation can then be found by simply integrating both sides with respect to each respective variable:

$$\int N(y)dy = -\int M(x)dx + C\,.$$

This is the *implicit* general solution of the equation, where y is defined implicitly as a function of x by the above equation relating the antiderivatives, with respect to their individual variables, of $M(x)$ and $N(y)$.

An *explicit* general solution, in the form of $y = f(x)$, where y is explicitly defined by a function $f(x)$ which itself satisfies the original differential equation, could be found (in theory, although not always in practice) by simplifying the implicit solution and solve for y.

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \qquad \leftarrow \text{laws of exponents}$$

$$= |1 + x|\,e^{c_1}$$

$$= \pm e^{c_1}(1 + x). \qquad \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases}$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(4) = -3$.

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. This solution of the differential equation represents a family of concentric circles centered at the origin.

Exercises

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\frac{dy}{dx} = \sin 5x$
2. $\frac{dy}{dx} = (x + 1)^2$
3. $dx + e^{3x}dy = 0$
4. $dy - (y - 1)^2 dx = 0$
5. $x\frac{dy}{dx} = 4y$
6. $\frac{dy}{dx} + 2xy^2 = 0$
7. $\frac{dy}{dx} = e^{3x+2y}$
8. $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$
9. $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$
10. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$
11. $\csc y\, dx + \sec^2 x\, dy = 0$
12. $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$
13. $(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy = 0$
14. $x(1 + y^2)^{1/2}\, dx = y(1 + x^2)^{1/2}\, dy$
15. $\frac{dS}{dr} = kS$
16. $\frac{dQ}{dt} = k(Q - 70)$
17. $\frac{dP}{dt} = P - P^2$
18. $\frac{dN}{dt} + N = Nte^{t+2}$
19. $\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$
20. $\frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$