Solve first order differential equations by exact method

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left hand side is an exact differential.

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (4)$$

Summary: Exact Equations

$$M(x,y) + N(x,y)\,y' = 0$$

Where there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y) .$$

1. Verification of exactness: it is an exact equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} .$$

2. The general solution is simply

$$\psi(x,y) = C.$$

Where the function $\psi(x,y)$ can be found by combining the result c two integrals (write down each distinct term only once, even if it appears in both integrals):

$$\psi(x,y) = \int M(x,y)\,dx , \quad \text{and}$$

$$\psi(x,y) = \int N(x,y)\,dy .$$

Example: Solve the equation

$$(y^4 - 2) + 4xy^3 y' = 0$$

First identify that $M(x,y) = y^4 - 2$, and $N(x,y) = 4xy^3$.

Then make sure that it is indeed an exact equation:

$$\frac{\partial M}{\partial y} = 4y^3 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 4y^3$$

Finally find $\psi(x,y)$ using partial integrations. First, we integrate M with respect to x. Then integrate N with respect to y.

$$\psi(x,y) = \int M(x,y)\,dx = \int (y^4 - 2)\,dx = xy^4 - 2x + C_1(y) \text{ ,}$$

$$\psi(x,y) = \int N(x,y)\,dy = \int 4xy^3\,dy = xy^4 + C_2(x) \text{ .}$$

Combining the result, we see that $\psi(x,y)$ must have 2 non-constant terms: xy^4 and $-2x$. That is, the (implicit) general solution is: $xy^4 - 2x = C$.

Now suppose there is the initial condition $y(-1) = 2$. To find the (implicit) particular solution, all we need to do is to substitute $x = -1$ and $y = 2$ into the general solution. We then get $C = -14$.

Therefore, the particular solution is $xy^4 - 2x = -14$.

Example: Solve the initial value problem

$$(y\cos(xy)+\frac{y}{x}+2x)\,dx+(x\cos(xy)+\ln x+e^{y})\,dy=0\,,\quad y(1)=($$

First, we see that $M(x,y)=y\cos(xy)+\frac{y}{x}+2x$ and $N(x,y)=x\cos(xy)+\ln x+e^{y}$.

Verifying:

$$\frac{\partial M}{\partial y}=-xy\sin(xy)+\cos(xy)+\frac{1}{x}=\frac{\partial N}{\partial x}=-xy\sin(xy)+\cos(xy)+\frac{1}{x}$$

Integrate to find the general solution:

$$\psi(x,y)=\int\left(y\cos(xy)+\frac{y}{x}+2x\right)dx=\sin(xy)+y\ln x+x^{2}+C_{1}(y)$$

as well,

$$\psi(x,y)=\int\left(x\cos(xy)+\ln x+e^{y}\right)dy=\sin(xy)+y\ln x+e^{y}+C_{2}(x)$$

Hence, $\sin xy+y\ln x+e^{y}+x^{2}=C.$

Apply the initial condition: $x=1$ and $y=0$:

$$C=\sin 0+0\ln(1)+e^{0}+1=2$$

The particular solution is then $\sin xy+y\ln x+e^{y}+x^{2}=2.$

Exercises

1–9. Determine if the equation is exact, and if it is exact, find the general solution.

1. $(y^2 + 2t) + 2tyy' = 0$
2. $y - t + (t + 2y)y' = 0$
3. $2t^2 - y + (t + y^2)y' = 0$
4. $y^2 + 2tyy' + 3t^2 = 0$
5. $(3y - 5t) + 2yy' - ty' = 0$
6. $2ty + (t^2 + 3y^2)y' = 0,\ y(1) = 1$
7. $2ty + 2t^3 + (t^2 - y)y' = 0$
8. $t^2 - y - ty' = 0$
9. $(y^3 - t)y' = y$