Second order linear differentail eqyution with constant coeffitions

>>Sluton of Non-homogeniuos SODEs with constant coeffitions

Theroem: The general solution of the second order nonhomogeneous linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be expressed in the form

$$y = y_c + Y$$

where Y is *any* specific function that satisfies the nonhomogeneous equation, and $y_c = C_1 y_1 + C_2 y_2$ is a general solution of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

(That is, y_1 and y_2 are a pair of fundamental solutions of the corresponding homogeneous equation; C_1 and C_2 are arbitrary constants.)

The term $y_c = C_1 y_1 + C_2 y_2$ is called the *complementary solution* (or the *homogeneous solution*) of the nonhomogeneous equation. The term Y is called the *particular solution* (or the *nonhomogeneous solution*) of the same equation.

solution of non-homogenious SODEs with constant coefficients (a,b,c) using udetermined coefficients method

SUMMARY: Method of Undetermined Coefficients

Given $\quad ay'' + by' + cy = g(t)$

1. Find the complementary solution y_c.

2. Subdivide, if necessary, $g(t)$ into parts: $g(t) = g_1(t) + g_2(t) \ldots + g_k(t)$.

3. For each $g_i(t)$, choose the form of its corresponding particular solution $Y_i(t)$ according to:

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)\, e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)\, e^{at}$
$P_n(t)\, e^{at}\cos\mu t$ and/or $P_n(t)\, e^{at}\sin\mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)\, e^{at}\cos\mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)\, e^{at}\sin\mu t$

Where $s = 0$, 1, or 2, is the **minimum** number of times the choice must be multiplied by t so that it shares no common terms with y_c.

$P_n(t)$ denotes a n-th degree polynomial. If there is no power of t present, then $n = 0$ and $P_0(t) = C_0$ is just the constant coefficient. If no exponential term is present, then set the exponent $a = 0$.

4. $Y = Y_1 + Y_2 + \ldots + Y_k$.

5. The general solution is $y = y_c + Y$.

6. Finally, apply any initial conditions to determine the as yet unknown coefficients C_1 and C_2 in y_c.

Example: $y'' - 2y' - 3y = e^{2t}$

The corresponding homogeneous equation $y'' - 2y' - 3y = 0$ has characteristic equation $r^2 - 2r - 3 = (r + 1)(r - 3) = 0$. So the complementary solution is $y_c = C_1 e^{-t} + C_2 e^{3t}$.

The nonhomogeneous equation has $g(t) = e^{2t}$. It is an exponent function, which does not change form after differentiation: an exponential function's derivative will remain an exponential fu with the same exponent (although its coefficient might change the effect of the Chain Rule). Therefore, we can very reasonab expect that $Y(t)$ is in the form Ae^{2t} for some unknown coefficie: Our job is to find this as yet undetermined coefficient.

Let $Y = Ae^{2t}$, then $Y' = 2Ae^{2t}$, and $Y'' = 4Ae^{2t}$. Substitute tl back into the original differential equation:

$$(4Ae^{2t}) - 2(2Ae^{2t}) - 3(Ae^{2t}) = e^{2t}$$

$$-3Ae^{2t} = e^{2t}$$

$$A = -1/3$$

Hence, $Y(t) = \dfrac{-1}{3} e^{2t}$.

Therefore, $y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - \dfrac{1}{3} e^{2t}$.

Example: $y'' - 2y' - 3y = 3t^2 + 4t - 5$

The corresponding homogeneous equation is still $y'' - 2y' - 3y = 0$. Therefore, the complementary solution remains $y_c = C_1 e^{-t} + C_2 e^{3t}$.

Now $g(t) = 3t^2 + 4t - 5$. It is a degree 2 (i.e., quadratic) polynomial. Since polynomials, like exponential functions, do not change form after differentiation: the derivative of a polynomial is just another polynomial of one degree less (until it eventually reaches zero). We expect that $Y(t)$ will, therefore, be a polynomial of the same degree as that of $g(t)$. (Why will their degrees be the same?)

So, we will let Y be a generic quadratic polynomial: $Y = At^2 + Bt + C$. It follows $Y' = 2At + B$, and $Y'' = 2A$. Substitute them into the equation:

$$(2A) - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 + 4t - 5$$

$$-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) = 3t^2 + 4t - 5$$

The corresponding terms on both sides should have the same coefficients, therefore, equating the coefficients of like terms.

t^2 :	$3 = -3A$			$A = -1$
t :	$4 = -4A - 3B$		$\rightarrow$	$B = 0$
1 :	$-5 = 2A - 2B - 3C$			$C = 1$

Therefore, $Y = -t^2 + 1$, and $y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - t^2 + 1$.

Exercises:

1 – 10 Find the general solution of each nonhomogeneous equation.

1. $y'' + 4y = 8$

2. $y'' + 4y = 8t^2 - 20t + 8$

3. $y'' + 4y = 5\sin 3t - 5\cos 3t$

4. $y'' + 4y = 24e^{-2t}$

5. $y'' + 4y = 8\cos 2t$

6. $y'' + 2y' = 2te^{-t}$

7. $y'' + 2y' = 6e^{-2t}$

8. $y'' + 2y' = 12t^2$

9. $y'' - 6y' - 7y = 13\cos 2t + 34\sin 2t$

10. $y'' - 6y' - 7y = 8e^{-t} - 7t - 6$