Higher Order Linear Equations with Constant Coefficients

The solutions of linear differential equations with constant coefficients of the third order or higher can be found in similar ways as the solutions of second order linear equations. For an *n*-th order homogeneous linear equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad a_n \neq 0.$$

It has a general solution of the form

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_{n-1} y_{n-1} + C_n y_n$$

Such a set of linearly independent solutions, and therefore, a general solutic of the equation, can be found by first solving the differential equation's characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 = 0.$$

This is a polynomial equation of degree n, therefore, it has n real and/or complex roots (not necessarily distinct). Those necessary n linearly independent solutions can then be found using the four rules below.

(i). If r is a distinct real root, then $y = e^{rt}$ is a solution.

(ii). If $r = \lambda \pm \mu i$ are distinct complex conjugate roots, then $y = e^{\lambda t} \cos \mu t$ and $y = e^{\lambda t} \sin \mu t$ are solutions.

(iii). If r is a real root appearing k times, then $y = e^{rt}$, $y = te^{rt}$, $y = t^2 e^{rt}$, … , and $y = t^{k-1} e^{rt}$ are all solutions.

(iv). If $r = \lambda \pm \mu i$ are complex conjugate roots each appears k times, then

$$y = e^{\lambda t} \cos \mu t, \quad y = e^{\lambda t} \sin \mu t,$$
$$y = te^{\lambda t} \cos \mu t, \quad y = te^{\lambda t} \sin \mu t,$$
$$y = t^2 e^{\lambda t} \cos \mu t, \quad y = t^2 e^{\lambda t} \sin \mu t,$$
$$\vdots \qquad \vdots$$
$$y = t^{k-1} e^{\lambda t} \cos \mu t, \text{ and } y = t^{k-1} e^{\lambda t} \sin \mu t,$$

are all solutions.

Example: $y^{(4)} - y = 0$

The characteristic equation is $r^4 - 1 = (r^2 + 1)(r + 1)(r - 1) = 0$, which has roots $r = 1, -1, i, -i$. Hence, the general solution is

$$y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t.$$

Example: $y^{(5)} - 3y^{(4)} + 3y^{(3)} - y'' = 0$

The characteristic equation is $r^5 - 3r^4 + 3r^3 - r^2 = r^2(r - 1)^3 = 0$, which has roots $r = 0$ (a double root), and 1 (a triple root). Hence, the general solution is

$$y = C_1 e^{0t} + C_2 t e^{0t} + C_3 e^t + C_4 t e^t + C_5 t^2 e^t$$

$$= C_1 + C_2 t + C_3 e^t + C_4 t e^t + C_5 t^2 e^t.$$

Example: $y^{(4)} + 4y^{(3)} + 8y'' + 8y' + 4y = 0$

The characteristic equation is $r^4 + 4r^3 + 8r^2 + 8r + 4 = (r^2 + 2r + 2)^2 = 0$, which has roots $r = -1 \pm i$ (repeated). Hence, the general solution is

$$y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t + C_3 t e^{-t} \cos t + C_4 t e^{-t} \sin t.$$

1 – 8 Find the general solution of each equation.

1. $y^{(3)} + 25y' = 0$

2. $y^{(3)} + 27y = 0$

3. $y^{(4)} - 18y'' + 81y = 0$

4. $y^{(4)} - 3y'' - 4y = 0$

5. $y^{(4)} + 32y'' + 256y = 0$

6. $y^{(5)} + 5y^{(4)} + 10y^{(3)} + 10y'' + 5y' + y = 0$

7. $y^{(5)} + 2y^{(4)} + 5y^{(3)} = 0$

8. $y^{(6)} - y = 0$