# Second Order Linear Partial Differential Equations

# Part I

*Second linear partial differential equations; Separation of Variables; 2-point boundary value problems; Eigenvalues and Eigenfunctions*

## Introduction

We are about to study a simple type of partial differential equations (PDEs): the second order linear PDEs. Recall that a partial differential equation is any differential equation that contains two or more independent variables. Therefore the derivative(s) in the equation are partial derivatives. We will examine the simplest case of equations with 2 independent variables. A few examples of second order linear PDEs in 2 variables are:

$$\alpha^2 u_{xx} = u_t \qquad \text{(one-dimensional heat conduction equation)}$$

$$a^2 u_{xx} = u_{tt} \qquad \text{(one-dimensional wave equation)}$$

$$u_{xx} + u_{yy} = 0 \qquad \text{(two-dimensional Laplace/potential equation)}$$

In this class we will develop a method known as the method of Separation of Variables to solve the above types of equations.

Hence, what we have is a problem given by:

| | | |
|---|---|---|
| (Heat conduction eq.) | $\alpha^2 u_{xx} = u_t$ , | $0 < x < L, \quad t > 0,$ |
| (Boundary conditions) | $u(0, t) = 0$, and | $u(L, t) = 0,$ |
| (Initial condition) | $u(x, 0) = f(x).$ | |

This is an example of what is known, formally, as an *initial-boundary value problem*. Although it is still true that we will find a general solution first, then apply the initial condition to find the particular solution. A major difference now is that the general solution is dependent not only on the equation, but also on the boundary conditions. In other words, the given partial differential equation will have different general solutions when paired with different sets of boundary conditions.

## Separation of Variables

Start with the one-dimensional heat conduction equation $\alpha^2 u_{xx} = u_t$ . Suppose that its solution $u(x, t)$ is such a function that it can be expressed as a product, $u(x, t) = X(x)T(t)$, where $X$ is a function of $x$ alone and $T$ is a function of $t$ alone. Then, its partial derivatives can also be expressed simply by:

| | |
|---|---|
| $u = XT$ | $u_{xx} = X''T$ |
| $u_x = X'T$ | $u_{tt} = XT''$ |
| $u_t = XT'$ | $u_{xt} = u_{tx} = X'T'$ |

Hence, the heat conduction equation $\alpha^2 u_{xx} = u_t$ can be rewritten as

$$\alpha^2 X'' T = X T'.$$

Dividing both sides by $\alpha^2 X T$:

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T}$$

Next, equate first the $x$-term and then the $t$-term with $-\lambda$. We have

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' = -\lambda X \quad \longrightarrow \quad X'' + \lambda X = 0,$$

and,

$$\frac{T'}{\alpha^2 T} = -\lambda \quad \rightarrow \quad T' = -\alpha^2 \lambda T \quad \longrightarrow \quad T' + \alpha^2 \lambda T = 0.$$

Consequently, the single partial differential equation has now been separated into a simultaneous system of 2 ordinary differential equations. They are a second order homogeneous linear equation in terms of $x$, and a first order linear equation (it is also a separable equation) in terms of $t$. Both of them can be solved easily using what we have already learned in this class.

*Example*: Separate $t^3 u_{xx} + x^3 u_{tt} = 0$ into an equation of $x$ and an equation of $t$.

Let $u(x,t) = X(x)T(t)$ and rewrite the equation in terms of $X$ and $T$:

$$t^3 X'' T + x^3 X T'' = 0,$$

$$t^3 X'' T = -x^3 X T''.$$

Divide both sides by $X''\, T''$, we have separated the variables:

$$\frac{t^3 T}{T''} = \frac{-x^3 X}{X''} \,.$$

Now insert a constant of separation:

$$\frac{t^3 T}{T''} = \frac{-x^3 X}{X''} = -\lambda \,.$$

Finally, rewrite it into 2 equations:

$$t^3 T = -\lambda T'' \qquad \rightarrow \qquad \lambda T'' + t^3 T = 0,$$
$$-x^3 X = -\lambda X'' \qquad \rightarrow \qquad \lambda X'' - x^3 X = 0.$$

*Example*: Separate

$$u_x + 2u_{tx} - 10u_{tt} = 0, \qquad u(0, t) = 0, \ \ u_x(L, t) = 0.$$

Let $u(x,t) = X(x)T(t)$ and rewrite the equation in terms of $X$ and $T$:

$$X'T + 2X'T' - 10XT'' = 0,$$

$$X'T + 2X'T' = 10XT''.$$

Divide both sides by $X'\,T''$, and insert a constant of separation:

$$\frac{T + 2T'}{T''} = \frac{10X}{X'} = -\lambda .$$

Rewrite it into 2 equations:

$$T + 2T' = -\lambda T'' \quad \rightarrow \quad \lambda T'' + 2T' + T = 0,$$
$$10X = -\lambda X' \quad \rightarrow \quad \lambda X' + 10X = 0.$$

The boundary conditions also must be separated:

$$u(0,t) = 0 \ \rightarrow \ X(0)T(t) = 0 \ \rightarrow \quad X(0) = 0 \quad \text{or} \quad T(t) = 0$$
$$u_x(L,t) = 0 \ \rightarrow \ X'(L)T(t) = 0 \ \rightarrow \quad X'(L) = 0 \quad \text{or} \quad T(t) = 0$$

As before, setting $T(t) = 0$ would result in the constant zero solution only. Therefore, we must choose the two (nontrivial) conditions in terms of $x$: $X(0) = 0$, and $X'(L) = 0$.