Undamped One-Dimensional Wave Equation: Vibrations of an Elastic String

Consider a piece of thin flexible string of length L, of negligible weight. Suppose the two ends of the string are firmly secured ("clamped") at some supports so they will not move. Assume the set-up has no damping. Then, the vertical displacement of the string, $0 < x < L$, and at any time $t > 0$, is given by the displacement function $u(x, t)$. It satisfies the *homogeneous one-dimensional undamped wave equation*:

$$a^2 u_{xx} = u_{tt}$$

Where the constant coefficient a^2 is given by the formula $a^2 = T/\rho$, such that a = horizontal propagation speed (also known as *phase velocity*) of the wave motion, T = force of tension exerted on the string, ρ = mass density (mass per unit length). It is subjected to the homogeneous boundary conditions

$$u(0, t) = 0, \text{ and } \quad u(L, t) = 0, \quad t > 0.$$

The two boundary conditions reflect that the two ends of the string are clamped in fixed positions. Therefore, they are held motionless at all time.

The equation comes with 2 initial conditions, due to the fact that it contains the second partial derivative of time, u_{tt}. The two initial conditions are the initial (vertical) displacement $u(x, 0)$, and the initial (vertical) velocity $u_t(x, 0)^*$, both are arbitrary functions of x alone. (Note that the string is merely the medium for the wave, it does not itself move horizontally, it only vibrates, vertically, in place. The resulting undulation, or the wave-like "shape" of the string, is what moves horizontally.)

Hence, what we have is the following initial-boundary value problem:

(Wave equation)	$a^2 u_{xx} = u_{tt}$,	$0 < x < L, \quad t > 0,$
(Boundary conditions)	$u(0, t) = 0$, and	$u(L, t) = 0,$
(Initial conditions)	$u(x, 0) = f(x)$, and	$u_t(x, 0) = g(x).$

We first let $u(x, t) = X(x)T(t)$ and separate the wave equation into two ordinary differential equations. Substituting $u_{xx} = X''T$ and $u_{tt} = XT''$ into the wave equation, it becomes

$$a^2 X'' T = XT''.$$

Dividing both sides by $a^2 X T$:

$$\frac{X''}{X} = \frac{T''}{a^2 T}$$

As for the heat conduction equation, it is customary to consider the constant a^2 as a function of t and group it with the rest of t-terms. Insert the constant of separation and break apart the equation:

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' = -\lambda X \quad \rightarrow \quad X'' + \lambda X = 0,$$

Example: Solve the one-dimensional wave problem

$$9u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0,t) = 0, \text{ and } \qquad u(5,t) = 0,$$
$$u(x,0) = 0,$$
$$u_t(x,0) = 4.$$

As in the previous example, $a^2 = 9$ (so $a = 3$), and $L = 5$. Therefore, the general solution remains

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{3n\pi t}{5} + B_n \sin \frac{3n\pi t}{5} \right) \sin \frac{n\pi x}{5}.$$

Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$\begin{aligned} &a^2 u_{xx} = u_{tt}\ , && 0 < x < L, \quad t > 0, \\ &u(0,t) = 0, \text{ and} && u(L,t) = 0, \\ &u(x,0) = f(x), \text{ and} && u_t(x,0) = g(x). \end{aligned}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L} \right) \sin\frac{n\pi x}{L}\ .$$

The particular solution can be found by the formulas:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\ , \qquad \text{and}$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\ .$$

1 – 4 Determine whether each PDE can be separated. Then separate it into two ODEs if it is possible to separate.

1. $x^2 u_{xx} - t^2 u_{tt} = 0$

2. $x\, u_{xx} - \pi\, u_{tt} = 5\, u_{xt}$

3. $u_{xx} - 3u = u_t\,,\qquad u(0, t) = 0,\quad u(\pi, t) = 0.$

4. $u_{xx} + 2t\, u_{tx} = 4\, u,\qquad u_x(0, t) = 0,\quad u(9, t) = 0.$

1. Solve the vibrating string problem of the given initial conditions.

$$4\, u_{xx} = u_{tt}\,,\qquad 0 < x < \pi,\quad t > 0,$$
$$u(0, t) = 0,\qquad u(\pi, t) = 0,$$

(a) $u(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x),$
$u_t(x, 0) = 0.$

(b) $u(x, 0) = 0,$
$u_t(x, 0) = 6.$

(c) $u(x, 0) = 0,$
$u_t(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x).$

2. Solve the vibrating string problem.

$$100\, u_{xx} = u_{tt}\,,\qquad 0 < x < 2,\quad t > 0,$$
$$u(0, t) = 0,\quad \text{and}\quad u(2, t) = 0,$$
$$u(x, 0) = 32\sin(\pi x) + e^2 \sin(3\pi x) + 25\sin(6\pi x),$$
$$u_t(x, 0) = 6\sin(2\pi x) - 16\sin(5\pi x / 2).$$

3. Solve the vibrating string problem.

$$25\, u_{xx} = u_{tt}\,,\qquad 0 < x < 1,\quad t > 0,$$
$$u(0, t) = 0,\quad \text{and}\quad u(2, t) = 0,$$
$$u(x, 0) = x - x^2,$$
$$u_t(x, 0) = \pi.$$