Solution of 2 × 2 systems of first order linear equations

Consider a system of 2 simultaneous first order linear equations

$$x_1' = a x_1 + b x_2$$
$$x_2' = c x_1 + d x_2$$

It has the alternate matrix-vector representation

$$\boldsymbol{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{x}.$$

Or, in shorthand $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, if $\boldsymbol{A}$ is already known from context.

We know that the above system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. Therefore, assume that $\boldsymbol{x} = \boldsymbol{k} e^{rt}$ is a solution of the system, where $\boldsymbol{k}$ is a vector of coefficients (of x_1 and x_2). Substitute $\boldsymbol{x}$ and $\boldsymbol{x}' = r\boldsymbol{k} e^{rt}$ into the equation $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, and we have

$$r\boldsymbol{k} e^{rt} = \boldsymbol{A}\boldsymbol{k} e^{rt}.$$

The possibilities are that A has

I. Two distinct real eigenvalues
II. Complex conjugate eigenvalues
III. A repeated eigenvalue

Case I Distinct real eigenvalues

If the coefficient matrix A has two distinct real eigenvalues r_1 and r_2, and their respective eigenvectors are $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2×2 system $\boldsymbol{x}' = A\boldsymbol{x}$ has a general solution

$$x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}.$$

We have already found that the coefficient matrix has eigenvalues $r = -1$ and 6. And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, a general solution of this system of differential equations is

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

Example: $\quad x' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The characteristic equation is $r^2 - r - 2 = (r+1)(r-2) = 0$. The eigenvalues are $r = -1$ and 2. They have, respectively, eigenvectors

For $r = -1$, the system is

$$(A - rI)\,x = (A + I)\,x = \begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} x = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the bottom equation of the system: $2x_1 - x_2 = 0$, we get the relation $x_2 = 2x_1$. Hence,

$$k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

For $r = 2$, the system is

$$(A - rI)\,x = (A - 2I)\,x = \begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} x = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $x_1 - 2x_2 = 0$, we get the relation $x_1 = 2x_2$. Hence,

$$k_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

Apply the initial values,

$$x(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0} = \begin{bmatrix} C_1 + 2C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

That is

$$\begin{aligned} C_1 + 2C_2 &= 1 \\ 2C_1 + C_2 &= -1 \end{aligned}.$$

We find $C_1 = -1$ and $C_2 = 1$, hence we have the particular solution

$$x = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-t} + 2e^{2t} \\ -2e^{-t} + e^{2t} \end{bmatrix}.$$

Case II Complex conjugate eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct complex conjugate eigenvalues $\lambda \pm \mu i$. Also suppose $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$ is an eigenvector (necessarily has complex-valued entries) of the eigenvalue $\lambda + \mu i$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a real-valued general solution

$$x = C_1 e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big) + C_2 e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}$$

The characteristic equation is $r^2 + 1 = 0$, giving eigenvalues $r = \pm i$. That is, $\lambda = 0$ and $\mu = 1$.

Take the first (the one with positive imaginary part) eigenvalue $r = i$, and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $(2-i)x_1 - 5x_2 = 0$, we get the relation $(2-i)x_1 = 5x_2$. Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{a} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{b} i = a + bi$$

Therefore, a general solution is

$$x = C_1 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(t)\right) + C_2 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(t)\right)$$

$$= C_1 \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + C_2 \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

<u>Case III</u> Repeated real eigenvalue

Suppose the coefficient matrix A has a repeated real eigenvalues r, there are 2 sub-cases.

(i) If r has two linearly independent eigenvectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalue is $r = 2$ (repeated). There are 2 sets of linearly independent eigenvectors, <u>which could be represented by any 2 nonzero vectors that are not constant multiples of each other</u>. For example

$$k_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{2t}.$$