Convolution Theorem

Statement: If $L^{-1}\{f(s)\}=f(t)$ and $L^{-1}\{g(s)\}=g(t)$ then

$$L^{-1}\{f(s)\,g(s)\}=\int_0^t f(u)g(t-u)du$$

Proof: Let $\varphi(t)=\int_0^t f(u)g(t-u)du$

Taking Laplace transform both sides, we get

$$L\{\varphi(t)\}=L\{\int_0^t f(u)g(t-u)du\}$$

$$=\int_0^\infty e^{-st}\int_0^t f(u)g(t-u)du\;dt$$

$$L\{\varphi(t)\}=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\;dt$$

$$=\int_{u=0}^\infty\int_{t=u}^\infty e^{-st}f(u)g(t-u)dt\;du$$

$$Put\; t-u=v\;\; then\;\; dt=dv,$$

$$t=u, v=0\;\; and\;\; t=\infty, v=\infty$$

$$L\{\varphi(t)\}=f(s)g(s)$$

$$L^{-1}\{f(s)g(s)\}=\varphi(t)$$

$$L^{-1}\{f(s)g(s)\}=\int_0^t f(u)g(t-u)du$$

Examples:

1. Using convolution theorem evaluate $L^{-1}\{\frac{1}{(s-1)(s^2+1)}\}$

Solution: Let $f(s) = \frac{1}{(s^2+1)}$ and $g(s) = \frac{1}{s-1}$

Therefore $L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = sint = f(t)$

$$L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{(s-1)}\right\} = e^t = g(t)$$

$$L^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s^2+1)} \cdot \frac{1}{s-1}\right\} = \int_0^t \sin u \, e^{t-u} du$$

$$= e^t \int_0^t \sin u \, e^{-u} du$$

$$= e^t [\frac{e^{-u}\{-sinu - cosu\}}{(-1)^2 + 1^2}]_0^t$$

$$= \frac{e^t}{2}[e^{-t}\{-sint - cost\} - 1\{-0-1\}]$$

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \frac{1}{2}[-sint - cot + e^t]$$

Using convolution theorem evaluate $L^{-1}\{\frac{1}{s(s^2+a^2)}\}$

Solution: Let $f(s) = \frac{1}{(s^2+a^2)}$ and $g(s) = \frac{1}{s}$

Therefore $L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\} = \frac{sinat}{a} = f(t)$

$$L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$$

$$L^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s^2+a^2)} \cdot \frac{1}{s}\right\} = \int_0^t \frac{\sin au}{a} 1du$$

$$= e^t \int_0^t \sin u\, e^{-u} du$$

$$= \frac{1}{a}[\frac{-\cos au}{a}]_0^t$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = -\frac{1}{a^2}[\cos at - 1]$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}[1 - \cos at]$$

Using convolution theorem evaluate $L^{-1}\{\frac{1}{s^2(s+5)}\}$

Solution: Let $f(s) = \frac{1}{(s+5)}$ and $g(s) = \frac{1}{s^2}$

Therefore $L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s+5)}\right\} = e^{-5t} = f(t)$

$$L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t)$$

$$L^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u)du$$

$$L^{-1}\left\{\frac{1}{s+5} \cdot \frac{1}{s^2}\right\} = \int_0^t e^{-5u}\,(t-u)du$$

Step Functions

Definition: The *unit step function* (or *Heaviside function*), is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}, \qquad c \geq 0.$$

Often the unit step function $u_c(t)$ is also denoted as $u(t-c)$, $H_c(t)$, or $H(t-c)$.

The step could also be made backward, stepping down from 1 to 0 at $t = c$. This complement function is

$$1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}, \qquad c \geq 0.$$

The Laplace transform of the unit step function is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \qquad s > 0, \qquad c \geq 0$$

Suppose

$$F(t)=\begin{cases} f_1(t), & t<a \\ f_2(t), & a\le t<b \\ f_3(t), & b\le t<c \\ \vdots & \vdots \\ f_n(t), & t\ge d \end{cases}.$$

Then, we can rewrite $F(t)$, succinctly, as

$$F(t) = (1 - u_a(t))f_1(t) + (u_a(t) - u_b(t))f_2(t) + (u_b(t) - u_c(t))f_3(t) + \ldots + u_d(t)f_n(t).$$

Example:

$$F(t)=\begin{cases} 3t^2-2, & t<4 \\ e^{5t}+t, & 4\le t<9 \\ \cos(2t), & t\ge 9 \end{cases}.$$

Then,

$$F(t) = (1 - u_4(t))\,(3t^2 - 2) + (u_4(t) - u_9(t))\,(e^{5t} + t) + u_9(t)\cos(2t).$$

Example: Find the Laplace transform of $u_2(t)\, e^{7t}$.

$$\mathcal{L}\{u_2(t)\, e^{7t}\} = e^{-2s}\, \mathcal{L}\{e^{7(t+2)}\} = e^{-2s}\, \mathcal{L}\{e^{7t+14}\} = e^{-2s}\, e^{14}\, \mathcal{L}$$

$$= e^{-2s} e^{14} \frac{1}{s-7} = \frac{e^{-2s+14}}{s-7}.$$

Example: Find the Laplace transform of $u_1(t)\,(t^2 + 3t + 2)$.

$$\mathcal{L}\{u_1(t)\,(t^2 + 3t + 2)\} = e^{-1s}\, \mathcal{L}\{(t+1)^2 + 3(t+1) + 2\} =$$

$$e^{-s}\, \mathcal{L}\{(t^2 + 2t + 1) + (3t + 3) + 2\} = e^{-s}\, \mathcal{L}\{t^2 + 5t + 6\}$$

$$= e^{-s}\left(\frac{2}{s^3} + 5\frac{1}{s^2} + 6\frac{1}{s}\right) = e^{-s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{6}{s}\right)$$

Exercises:

1. Find (a) $\mathcal{L}\{u_\pi(t)\, t^2\}$, (b) $\mathcal{L}\{u_4(t)\, t^2\, e^{5t}\}$.

2. Find (a) $\mathcal{L}\{u_{5\pi/6}(t) \cos 3t\}$, (b) $\mathcal{L}\{u_{\pi/2}(t)\, e^{-t} \cos 2t\}$.

3. Find $\mathcal{L}\{u_3(t)\,(t^2 - t + 2)\, e^{-5t}\}$.

4. Suppose $f(t) = \sin t + u_1(t) - 5u_4(t) - 2u_5(t) \cos t + \pi\, u_9(t)$, find $f(0)$, $f(\pi)$, $f(2\pi)$, and $f(8)$.

7. Find the Laplace transform of

$$F(t)=\begin{cases} t^2+t, & 0\le t<2 \\ 1-e^{-4t}, & 2\le t<5 \\ 0, & 5\le t \end{cases}.$$

8 – 13 Find the inverse Laplace transform of each given $F(s)$.

8. $F(s)=e^{-4s}\dfrac{3s+22}{s^2+3s-10}$

9. $F(s)=e^{-6s}\dfrac{4s+11}{s^2+6s+9}$

10. $F(s)=e^{-s}\dfrac{3s^3+12s^2-2s-3}{s^4-2s^3-3s^2}$

11. $F(s)=e^{-2s}\dfrac{2s-14}{s^2+2s+17}$

12. $F(s)=e^{-8s}\dfrac{3s^2-10s+8}{s^3+4s}$

13. $F(s)=\dfrac{e^{-cs}}{(s+\alpha)(s+\beta)}$

16 – 20 Solve each initial value problem.

16. $y'+6y=4u_2(t)\,t^2,\quad y(0)=1$

17. $y''+6y'+9y=u_5(t)\,e^{-t},\quad y(0)=10,\quad y'(0)=0$

18. $y''+4y'+5y=u_3(t)-u_6(t),\quad y(0)=0,\quad y'(0)=4$

19. $y''+5y'+4y=u_{10}(t)-2u_{20}(t),\quad y(0)=2,\quad y'(0)=0$

20. $y''+25y=t-t\,u_6(t),\quad y(0)=0,\quad y'(0)=3$