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Solving initial value problems using the method of Laplace transforms

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

1. Take the Laplace transforms of both sides of an equation.

2. Simplify algebraically the result to solve for $\mathcal{L}\{y\} = Y(s)$ in terms of s.

3. Find the inverse transform of $Y(s)$. (Or, rather, find a function $y(t)$ whose Laplace transform matches the expression of $Y(s)$.) This inverse transform, $y(t)$, is the solution of the given differential equation.

Example: $y'' - 6y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = -3$

[Step 1] Transform both sides

$$\mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{0\}$$

$$(s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)) - 6(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = 0$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2\mathcal{L}\{y\} - s - (-3)) - 6(s\,\mathcal{L}\{y\} - 1) + 5\mathcal{L}\{y\} = 0$$

$$(s^2 - 6s + 5)\mathcal{L}\{y\} - s + 9 = 0$$

$$(s^2 - 6s + 5)\mathcal{L}\{y\} = s - 9$$

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}$$

[Step 3] Find the inverse transform $y(t)$

Use partial fractions to simplify,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{a}{s-1} + \frac{b}{s-5}$$

$$\frac{s-9}{s^2 - 6s + 5} = \frac{a(s-5)}{(s-1)(s-5)} + \frac{b(s-1)}{(s-5)(s-1)}$$

$$s - 9 = a(s-5) + b(s-1) = (a+b)s + (-5a - b)$$

Equating the corresponding coefficients:

$$1 = a + b \qquad\qquad a = 2$$
$$-9 = -5a - b \qquad\qquad b = -1$$

Hence,

$$\mathscr{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}.$$

The last expression corresponds to the Laplace transform of $2e^t - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^t - e^{5t}.$$

Example: $\quad y' + 2y = 4te^{-2t}, \qquad y(0) = -3.$

[Step 1] Transform both sides

$$\mathscr{L}\{y' + 2y\} = \mathscr{L}\{4te^{-2t}\}$$

$$(s\mathscr{L}\{y\} - y(0)) + 2\mathscr{L}\{y\} = \mathscr{L}\{4te^{-2t}\} = \frac{4}{(s+2)^2}$$

[Step 2] Simplify to find $Y(s) = \mathscr{L}\{y\}$

$$(s\mathscr{L}\{y\} - (-3)) + 2\mathscr{L}\{y\} = \frac{4}{(s+2)^2}$$

$$(s + 2)\mathscr{L}\{y\} + 3 = \frac{4}{(s+2)^2}$$

$$(s + 2)\mathscr{L}\{y\} = \frac{4}{(s+2)^2} - 3$$

$$\mathscr{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{s+2} = \frac{4 - 3(s+2)^2}{(s+2)^3} = \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathscr{L}\{y\} = \frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{s+2}.$$

$$\frac{-3s^2-12s-8}{(s+2)^3}=\frac{a}{(s+2)^3}+\frac{b(s+2)}{(s+2)^3}+\frac{c(s+2)^2}{(s+2)^3}$$

$$=\frac{a+bs+2b+cs^2+4cs+4c}{(s+2)^3}=\frac{cs^2+(b+4c)s+(a+2b+4c)}{(s+2)^3}$$

$$-3=c \qquad a=4$$
$$-12=b+4c \qquad b=0$$
$$-8=a+2b+4c \qquad c=-3$$

$$\mathscr{L}\{y\}=\frac{-3s^2-12s-8}{(s+2)^3}=\frac{4}{(s+2)^3}-\frac{3}{s+2}.$$

This expression corresponds to the Laplace transform of $2t^2e^{-2t}-3e^{-2t}$. Therefore,

$$y(t)=2t^2e^{-2t}-3e^{-2t}.$$

Example: $y'' - 3y' + 2y = e^{3t}, \qquad y(0) = 1, \quad y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - s - 0) - 3(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} - s + 3 = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} = s - 3 + \frac{1}{s-3} = \frac{(s-3)^2 + 1}{s-3}$$

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s^2 - 3s + 2)(s-3)} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}.$$

Therefore, $y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$.

Example 11: Initial Value Problem (1 of 2)

- Consider the initial value problem

$$y'' - 8y' + 25y = 0, \quad y(0) = 0, \quad y'(0) = 6$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] - 8\left[sL\{y\} - y(0)\right] + 25L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 - 8s + 25\right)Y(s) - (s - 8)y(0) - y'(0) = 0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 - 8s + 25\right)Y(s) - 6 = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{6}{s^2 - 8s + 25}$$

Completing the square, we obtain

$$Y(s) = \frac{6}{s^2 - 8s + 25} = \frac{6}{\left(s^2 - 8s + 16\right) + 9}$$

Thus

$$Y(s) = 2\left[\frac{3}{(s - 4)^2 + 9}\right]$$

Using Table 6.2.1, we have

$$L^{-1}\{Y(s)\} = 2e^{4t}\sin 3t$$

Therefore our solution to the initial value problem is

$$y(t) = 2e^{4t}\sin 3t$$