7.2 Power Series Solutions About an Ordinary Point

A point t_0 is called an ***ordinary point*** of $Ly = 0$ if we can write the differential equation in the form

$$y'' + a_1(t)y' + a_0(t)y = 0, \qquad (1)$$

where $a_0(t)$ and $a_1(t)$ are analytic at t_0. If t_0 is not an ordinary point, we call it a ***singular point***.

Example 1. Determine the ordinary and singular points for each of the following differential equations:

1. $y'' + \frac{1}{t^2-9}y' + \frac{1}{t+1}y = 0.$
2. $(1-t^2)y'' - 2ty' + n(n+1)y = 0$, where n is an integer.
3. $ty'' + (\sin t)y' + (e^t - 1)y = 0.$

► **Solution.**

1. Here $a_1(t) = \frac{1}{t^2-9}$ is analytic except at $t = \pm 3$. The function $a_0 = \frac{1}{t+1}$ is analytic except at $t = -1$. Thus, the singular points are -3, 3, and -1. All other points are ordinary.
2. This is Legendre's equation. In standard form, we find $a_1(t) = \frac{-2t}{1-t^2}$ and $a_0(t) = \frac{n(n+1)}{1-t^2}$. They are analytic except at 1 and -1. These are the singular points and all other points are ordinary.
3. In standard form, $a_1(t) = \frac{\sin t}{t}$ and $a_0(t) = \frac{e^t-1}{t}$. Both of these are analytic

Suppose $a_0(t)$ and $a_1(t)$ are analytic at t_0, both of which converge for
Then there is a unique solution $y(t)$, analytic at t_0, to the initial value

$$y'' + a_1(t)y' + a_0(t)y = 0, \quad y(t_0) = \alpha, \; y'(t_0) = \beta. \tag{2}$$

$$y(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$$

Example 3. Use the power series method to solve

$$y'' + y = 0.$$

► **Solution.** Of course, this is a constant coefficient differential equation. Since $q(s) = s^2 + 1$ and $\mathcal{B}_q = \{\cos t, \sin t\}$, we get solution $y(t) = c_1 \sin t + c_2 \cos t$. Let us see how the power series method gives the same answer. Since the coefficients are constant, they are analytic everywhere with infinite radius of convergence. Theorem 2 implies that the power series solutions converge everywhere. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$ be a power series about $t_0 = 0$. Then

$$y'(t) = \sum_{n=1}^{\infty} c_n n t^{n-1}$$

$$\text{and} \qquad y''(t) = \sum_{n=2}^{\infty} c_n n(n-1)t^{n-2}.$$

An index shift, $n \to n + 2$, gives $y''(t) = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n$. Therefore, the equation $y'' + y = 0$ gives

$$\sum_{n=0}^{\infty} (c_n + c_{n+2}(n+2)(n+1))t^n = 0,$$

which implies $c_n + c_{n+2}(n+2)(n+1) = 0$, or, equivalently,

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)} \quad \text{for all } n = 0, 1, \ldots. \tag{3}$$

2, it follows that even terms are determined by previous even terms and odd terms are determined by previous odd terms. Let us consider these two cases separately.

The Even Case

$$\begin{aligned} n=0 &\quad c_2 = \tfrac{-c_0}{2\cdot 1} \\ n=2 &\quad c_4 = \tfrac{-c_2}{4\cdot 3} = \tfrac{c_0}{4\cdot 3\cdot 2\cdot 1} = \tfrac{c_0}{4!} \\ n=4 &\quad c_6 = \tfrac{-c_4}{6\cdot 5} = \tfrac{-c_0}{6!} \\ n=6 &\quad c_8 = \tfrac{-c_6}{8\cdot 7} = \tfrac{c_0}{8!} \\ \vdots &\quad \vdots \end{aligned}$$

More generally, we can see that

$$c_{2n} = (-1)^n \frac{c_0}{(2n)!}.$$

The Odd Case

$$\begin{aligned} n=1 &\quad c_3 = \tfrac{-c_1}{3\cdot 2} \\ n=3 &\quad c_5 = \tfrac{-c_3}{5\cdot 4} = \tfrac{c_1}{5\cdot 4\cdot 3\cdot 2} = \tfrac{c_1}{5!} \\ n=5 &\quad c_7 = \tfrac{-c_5}{7\cdot 6} = \tfrac{-c_1}{7!} \\ n=7 &\quad c_9 = \tfrac{-c_7}{9\cdot 8} = \tfrac{c_1}{9!} \\ \vdots &\quad \vdots \end{aligned}$$

Similarly, we see that

$$c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}.$$

Now, as we mentioned in Sect. 7.1, we can change the order of absolutely convergent sequences without affecting the sum. Thus, let us rewrite $y(t) = \sum_{n=0}^{\infty} c_n t^n$ in terms of odd and even indices to get

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_{2n} t^{2n} + \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} \\ &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \\ &= c_0 \cos t + c_1 \sin t, \end{aligned}$$

$$2t^2y'' + 5ty' - 2y = 0. \tag{5}$$

Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then

$$2t^2y'' = \sum_{n=0}^{\infty} 2n(n-1)c_n t^n,$$

$$5ty' = \sum_{n=0}^{\infty} 5nc_n t^n,$$

$$-2y = \sum_{n=0}^{\infty} -2c_n t^n.$$

Thus,

$$2t^2y'' + 5ty' - 2y = \sum_{n=0}^{\infty}(2n(n-1) + 5n - 2)c_n t^n = \sum_{n=0}^{\infty}(2n-1)(n+2)c_n t^n.$$

Find a series solution in powers of x of Airy's[4] equation

$$y'' - xy = 0, \qquad -\infty < x < \infty. \tag{15}$$

For this equation $P(x) = 1, Q(x) = 0$, and $R(x) = -x$; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{16}$$

and that the series converges in some interval $|x| < \rho$. The series for y'' is given by Eq. (7); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{17}$$

Substituting the series (16) and (17) for y and y'' in Eq. (15), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{18}$$

Next, we shift the index of summation in the series on the right side of Eq. (18) by replacing n by $n-1$ and starting the summation at 1 rather than zero. Thus we have

$$2 \cdot 1a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Again, for this equation to be satisfied for all x in some interval, the coefficients of like powers of x must be equal; hence $a_2 = 0$, and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1} \qquad \text{for} \quad n = 1, 2, 3, \ldots. \tag{19}$$

Since a_{n+2} is given in terms of a_{n-1}, the a's are determined in steps of three. Thus a_0 determines a_3, which in turn determines $a_6, \ldots$; a_1 determines a_4, which in turn determines $a_7, \ldots$; and a_2 determines a_5, which in turn determines $a_8, \ldots$. Since $a_2 = 0$, we immediately conclude that $a_5 = a_8 = a_{11} = \cdots = 0$.

For the sequence $a_0, a_3, a_6, a_9, \ldots$ we set $n = 1, 4, 7, 10, \ldots$ in the recurrence relation:

$$a_3 = \frac{a_0}{2 \cdot 3}, \qquad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \qquad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \ldots.$$

These results suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}, \qquad n \geq 4.$$

For the sequence $a_1, a_4, a_7, a_{10}, \ldots$, we set $n = 2, 5, 8, 11, \ldots$ in the recurrence relation:

$$a_4 = \frac{a_1}{3 \cdot 4}, \qquad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \qquad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \ldots.$$

In general, we have

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}, \qquad n \geq 4.$$

Thus the general solution of Airy's equation is

$$y = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right]$$

$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right].$$

PROBLEMS

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' + 3xy' + 5y = 0$
5. $x^2y'' - xy' + y = 0$
6. $(x-1)^2y'' + 8(x-1)y' + 12y = 0$
7. $x^2y'' + 6xy' - y = 0$
8. $2x^2y'' - 4xy' + 6y = 0$
9. $x^2y'' - 5xy' + 9y = 0$
10. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$
11. $x^2y'' + 2xy' + 4y = 0$
12. $x^2y'' - 4xy' + 4y = 0$