Laplace Transform

Definition (Laplace Transform)

Let f be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral,

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

The domain of $F(s)$ is the set of all values of s for which this integral converges. The Laplace transform of f is denoted by both F and $\mathcal{L}$.

Convention uses s as the independent variable and capital letters for the transformed functions:

$$\mathcal{L}[f] = F \qquad \mathcal{L}[y] = Y \qquad \mathcal{L}[x] = X$$

$$\mathcal{L}[f](s) = F(s) \qquad \mathcal{L}[y](s) = Y(s) \qquad \mathcal{L}[x](s) = X(s)$$

Examples: Laplace Transform

Example 1: Let $f(t) = 1, t \geq 0$. The Laplace transform satisfies:

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} dt = -\lim_{A\to\infty} \frac{e^{-st}}{s}\bigg|_0^A = -\lim_{A\to\infty} \left(\frac{e^{-sA}}{s} - \frac{1}{s}\right) = \frac{1}{s},$$

Example 2: Let $f(t) = e^{at}, t \geq 0$. The Laplace transform satisfies:

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}, \qquad s > a.$$

Example 3: Let $f(t) = e^{(a+bi)t}, t \geq 0$. The Laplace transform satisfies:

$$\mathcal{L}[e^{(a+bi)t}] = \int_0^{\infty} e^{-st} e^{(a+bi)t} dt = \int_0^{\infty} e^{-(s-a-bi)t} dt = \frac{1}{s-a-bi},$$

$s > a.$

Examples: Laplace Transform

Example 4: Let $f(t) = \sin(at), t \geq 0$. But

$$\sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right).$$

By linearity, the Laplace transform satisfies:

$$\mathcal{L}[\sin(at)] = \frac{1}{2i}\left(\mathcal{L}[e^{iat}] - \mathcal{L}[e^{-iat}]\right) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{a}{s^2+a^2}$$

$s > 0.$

Example 5: Let $f(t) = 2 + 5e^{-2t} - 3\sin(4t), t \geq 0$. By linearity, the Laplace transform satisfies:

$$\begin{aligned}\mathcal{L}[2 + 5e^{-2t} - 3\sin(4t)] &= 2\mathcal{L}[1] + 5\mathcal{L}[e^{-2t}] - 3\mathcal{L}[\sin(4t)] \\ &= \frac{2}{s} + \frac{5}{s+2} - \frac{12}{s^2+16}, \qquad s > 0.\end{aligned}$$

Laplace Transform - $e^{ct}f(t)$

Laplace Transform - $e^{ct}f(t)$: Previously found Laplace transforms of several basic functions

Theorem (Exponential Shift Theorem)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a$*, and if* c *is a constant, then*

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Example

Example: Consider the function

$$g(t) = e^{-2t}\cos(3t).$$

From our **Table of Laplace Transforms**, if $f(t) = \cos(3t)$, then

$$F(s) = \frac{s}{s^2+9}, \qquad s > 0.$$

From our previous theorem, the Laplace transform of $g(t)$ satisfies:

$$G(s) = \mathcal{L}[e^{-2t}f(t)] = F(s+2) = \frac{s+2}{(s+2)^2+9}, \qquad s > -2.$$

Laplace Transform of Derivatives

Theorem (Laplace Transform of Derivatives)

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Suppose that f and f' are of exponential order with $|f^{(i)}(t)| \leq Ke^{at}|$ for some constants K and a and $i = 0, 1$. Then $\mathcal{L}[f'(t)]$ exists for $s > a$, and moreover

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

For our 2^{nd} order differential equations we will commonly use

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

Laplace Transform of Derivatives - Example

Example: Consider

$$g(t) = e^{-2t}\sin(4t) \qquad \text{with} \qquad g'(t) = -2e^{-2t}\sin(4t) + 4e^{-2t}\cos(4t)$$

If $f(t) = \sin(4t)$, then

$$F(s) = \frac{4}{s^2 + 16}, \qquad \text{with} \qquad G(s) = \frac{4}{(s+2)^2 + 16}$$

using the exponential theorem of Laplace transforms

Our derivative theorem gives

$$\mathcal{L}[g'(t)] = sG(s) - g(0) = \frac{4s}{(s+2)^2 + 16}$$

Multiplication by t

let f be a signal and define

$$g(t) = tf(t)$$

then we have

$$G(s) = -F'(s)$$

to verify formula, just differentiate both sides of

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

with respect to s to get

$$F'(s) = \int_0^\infty (-t)e^{-st} f(t)\, dt$$

examples

- $f(t) = e^{-t}$, $g(t) = te^{-t}$

$$\mathcal{L}(te^{-t}) = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

- $f(t) = te^{-t}$, $g(t) = t^2e^{-t}$

$$\mathcal{L}(t^2e^{-t}) = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

- in general,

$$\mathcal{L}(t^ke^{-t}) = \frac{(k-1)!}{(s+1)^{k+1}}$$