Inverse Problem

- The main difficulty in using the Laplace transform method is determining the function $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.
- This is an inverse problem, in which we try to find ϕ such that $\phi(t) = L^{-1}\{Y(s)\}$.

Linearity of the Inverse Transform

- Frequently a Laplace transform $F(s)$ can be expressed as

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

- Let

$$f_1(t) = L^{-1}\{F_1(s)\}, \ldots, f_n(t) = L^{-1}\{F_n(s)\}$$

- Then the function

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$, since L is linear.
- By the uniqueness result of the previous slide, no other continuous function f has the same transform $F(s)$.
- Thus L^{-1} is a linear operator with

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + \cdots + L^{-1}\{F_n(s)\}$$

Example 2

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{2}{s}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{2}{s} = 2\left(\frac{1}{s}\right)$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{s}\right\} = 2L^{-1}\left\{\frac{1}{s}\right\} = 2(1) = 2$$

- Thus

$$y(t) = 2$$

Find the inverse Laplace Transform of the given function.

$$Y(s)=\frac{3}{s-5}$$

To find $y(t)$ such that $y(t)=L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s)=\frac{3}{s-5}=3\left(\frac{1}{s-5}\right)$$

Using Table 6.2.1,

$$L^{-1}\{Y(s)\}=L^{-1}\left\{\frac{3}{s-5}\right\}=3L^{-1}\left\{\frac{1}{s-5}\right\}=3e^{5t}$$

Thus

$$y(t)=3e^{5t}$$

Find the inverse Laplace Transform of the given function.

$$Y(s)=\frac{6}{s^4}$$

To find $y(t)$ such that $y(t)=L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s)=\frac{6}{s^4}=\frac{3!}{s^4}$$

Using Table 6.2.1,

$$L^{-1}\{Y(s)\}=L^{-1}\left\{\frac{3!}{s^4}\right\}=t^3$$

Thus

$$y(t)=t^3$$

Find the inverse Laplace Transform of the given function.

$$Y(s)=\frac{4s+1}{s^2+9}$$

To find $y(t)$ such that $y(t)=L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s)=\frac{4s+1}{s^2+9}=4\left[\frac{s}{s^2+9}\right]+\frac{1}{3}\left[\frac{3}{s^2+9}\right]$$

Using Table 6.2.1,

$$L^{-1}\{Y(s)\}=4L^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}=4\cos 3t+\frac{1}{3}\sin 3t$$

Thus

$$y(t)=4\cos 3t+\frac{1}{3}\sin 3t$$

For the function $Y(s)$ below, we find $y(t) = L^{-1}\{Y(s)\}$ by using a partial fraction expansion, as follows.

$$Y(s) = \frac{3s+1}{s^2+s-12} = \frac{3s+1}{(s+4)(s-3)} = \frac{A}{s+4} + \frac{B}{s-3}$$

$$3s+1 = A(s-3) + B(s+4)$$

$$3s+1 = (A+B)s + (4B-3A)$$

$$A+B=3, \quad 4B-3A=1$$

$$A = 11/7, \quad B = 10/7$$

$$Y(s) = \frac{11}{7}\left[\frac{1}{s+4}\right] + \frac{10}{7}\left[\frac{1}{s-3}\right] \Rightarrow y(t) = \frac{11}{7}e^{-4t} + \frac{10}{7}e^{3t}$$

For the function $Y(s)$ below, we find $y(t) = L^{-1}\{Y(s)\}$ by completing the square in the denominator and rearranging the numerator, as follows.

$$Y(s) = \frac{4s-10}{s^2-6s+10} = \frac{4s-10}{\left(s^2-6s+9\right)+1} = \frac{4s-12+2}{(s-3)^2+1}$$

$$= \frac{4(s-3)+2}{(s-3)^2+1} = 4\left[\frac{s-3}{(s-3)^2+1}\right] + 2\left[\frac{1}{(s-3)^2+1}\right]$$

Using Table 6.1, we obtain

$$y(t) = 4e^{3t}\cos t + 2e^{3t}\sin t$$

Partial fraction method

Use partial fractions to simplify,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2-6s+5} = \frac{a}{s-1} + \frac{b}{s-5}$$

$$\frac{s-9}{s^2-6s+5} = \frac{a(s-5)}{(s-1)(s-5)} + \frac{b(s-1)}{(s-5)(s-1)}$$

$$s-9 = a(s-5) + b(s-1) = (a+b)s + (-5a-b)$$

Equating the corresponding coefficients:

$$1 = a + b \qquad a = 2$$
$$-9 = -5a - b \qquad b = -1$$

Hence,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2-6s+5} = \frac{2}{s-1} - \frac{1}{s-5}.$$

The last expression corresponds to the Laplace transform of $2e^t - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^t - e^{5t}.$$