20/2/2020

Double Integrals Over General Regions

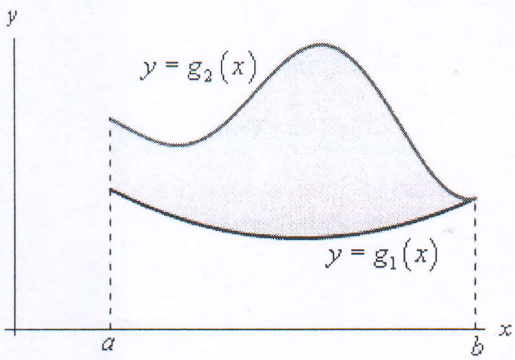
In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint_D f(x,y)\,dA$$

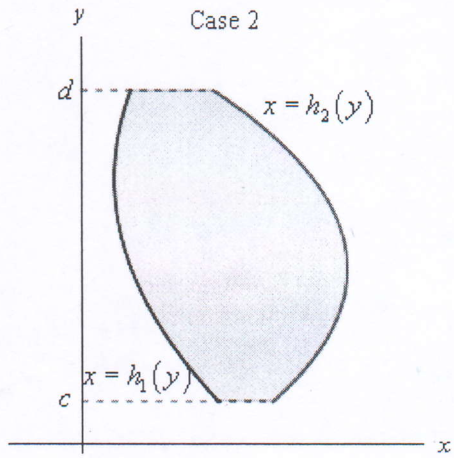
where D is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.

Case 1



Case 2



Case 1

$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Case 2.

$D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

In Case 1 where $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ the integral is defined to be,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

In Case 2 where $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$ the integral is defined to be,

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

ropert es

1. $\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

2. $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$, where c is any constant.

3. If the region D can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Example 1 Evaluate each of the following integrals over the given region D.

(a) $\iint_D e^{\frac{x}{y}}\, dA$, $D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$

(b) $\iint_D 4xy - y^3\, dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

(c) $\iint_D 6x^2 - 40y\, dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Solution

(a) $\iint_D e^{\frac{x}{y}}\, dA$, $D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$

Okay, this first one is set up to just use the formula above so let's do that.

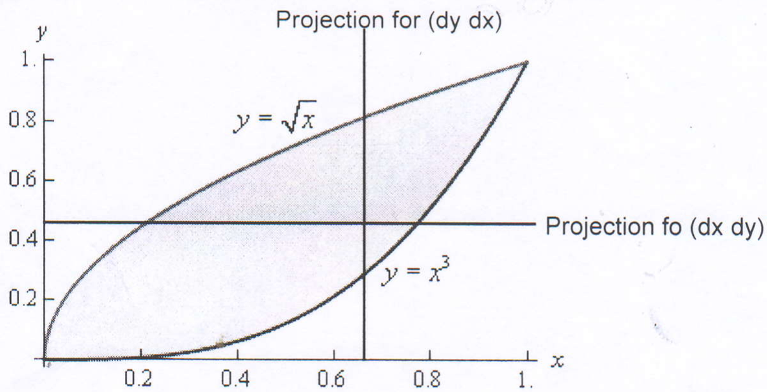
$$\iint_D e^{\frac{x}{y}}\, dA = \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\, dx\, dy = \int_1^2 \left. y e^{\frac{x}{y}} \right|_y^{y^3} dy$$

$$= \int_1^2 y e^{y^2} - y e^1\, dy$$

25/2

$$= \left(\frac{1}{2} e^{y^2} - \frac{1}{2} y^2 e^1 \right) \Bigg|_1^2 = \frac{1}{2} e^4 - 2e^1$$

(b) $\iint_D 4xy - y^3\, dA$, ***D* is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$.



So, from the sketch we can see that that two inequalities are,

$$0 \le x \le 1 \qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral,

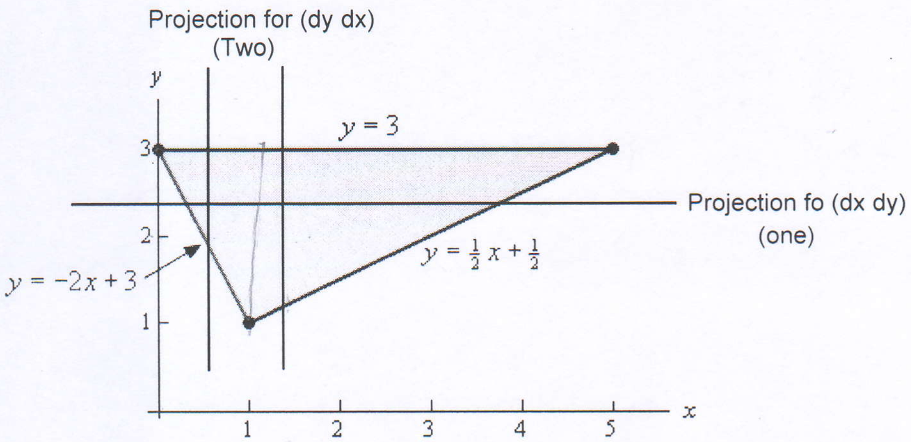
$$\iint_D 4xy - y^3\, dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx$$

$$= \int_0^1 \left(2xy^2 - \frac{1}{4} y^4 \right) \Bigg|_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{7}{4} x^2 - 2x^7 + \frac{1}{4} x^{12}\, dx$$

$$= \left(\frac{7}{12} x^3 - \frac{1}{4} x^8 + \frac{1}{52} x^{13} \right) \Bigg|_0^1 = \frac{55}{156}$$

(c) $\iint_D 6x^2 - 40y\, dA$, ***D* is the triangle with vertices** $(0,3)$, $(1,1)$, **and** $(5,3)$.



$$D_1 = \{(x, y) \mid 0 \le x \le 1,\ -2x+3 \le y \le 3\}$$

$$D_2 = \left\{(x, y) \mid 1 \le x \le 5,\ \frac{1}{2}x + \frac{1}{2} \le y \le 3\right\}$$

To avoid this we could turn things around and solve the two equations for x to get,

$$y = -2x + 3 \qquad \Rightarrow \qquad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad \Rightarrow \qquad x = 2y - 1$$

$$D = \left\{(x, y) \mid -\frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1,\ 1 \le y \le 3\right\}$$

Sdution 1

$$\iint_D 6x^2 - 40y\,dA = \iint_{D_1} 6x^2 - 40y\,dA + \iint_{D_2} 6x^2 - 40y\,dA$$

$$= \int_0^1 \int_{-2x+3}^{3} 6x^2 - 40y\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^{3} 6x^2 - 40y\,dy\,dx$$

$$= \int_0^1 \left(6x^2y - 20y^2\right)\Big|_{-2x+3}^{3} dx + \int_1^5 \left(6x^2y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3} dx$$

$$= \int_0^1 12x^3 - 180 + 20(3-2x)^2\,dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^2 dx$$

$$= \left(3x^4 - 180x - \tfrac{10}{3}(3-2x)^3\right)\Big|_0^1 + \left(-\tfrac{3}{4}x^4 + 5x^3 - 180x + \tfrac{40}{3}\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^3\right)\Big|_1^5$$

$$= -\frac{935}{3}$$

Sdution 2

This solution will be a lot less work since we are only going to do a single integral.

$$\iint_D 6x^2 - 40y\,dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} 6x^2 - 40y\,dx\,dy$$

$$= \int_1^3 \left(2x^3 - 40xy\right)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 100y - 100y^2 + 2(2y-1)^3 - 2\left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^3 dy$$

$$= \left(50y^2 - \tfrac{100}{3}y^3 + \tfrac{1}{4}(2y-1)^4 + \left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^4\right)\Big|_1^3$$

$$-\frac{935}{3}$$

Example 2 Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

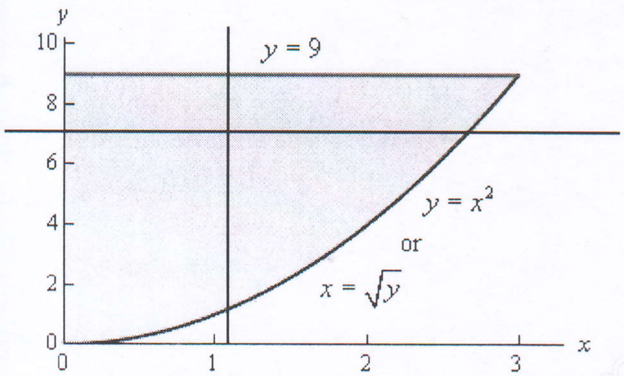
(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$

Solution

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

$0 \le x \le 3$ that mean x=0, x=3

$x^2 \le y \le 9$ that mean y=9, y=x^2 | from this value we draw the graph as shown



so the limits for dxdy

$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 9$$

The integral, with the order reversed, is now,

$$\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy$$

$$\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy$$

$$= \int_0^9 \frac{1}{4} x^4 \mathbf{e}^{y^3} \Big|_0^{\sqrt{y}}\, dy$$

$$= \int_0^9 \frac{1}{4} y^2 \mathbf{e}^{y^3}\, dy$$

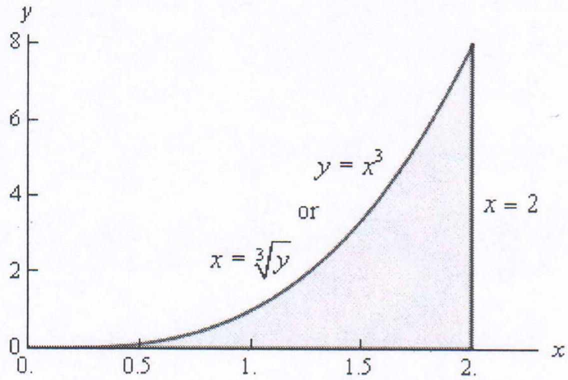
$$= \frac{1}{12} \mathbf{e}^{y^3} \Big|_0^9$$

$$= \frac{1}{12}\left(\mathbf{e}^{729} - 1\right)$$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$

$\sqrt[3]{y} \le x \le 2$ | That mean x = $\sqrt[3]{y}$,x=2 | so the graph will be

$0 \le y \le 8$ | That mean y=0 , y= 8



So, if we reverse the order of integration we get the following limits.

$$0 \le x \le 2$$
$$0 \le y \le x^3$$

The integral is then,

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy = \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\, dy\, dx$$
$$= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3} dx$$
$$= \int_0^2 x^3\sqrt{x^4+1}\, dx = \frac{1}{6}\left(17^{\frac{3}{2}} - 1\right)$$

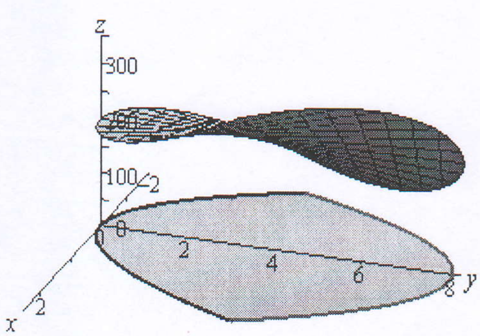
The Volume of Solid

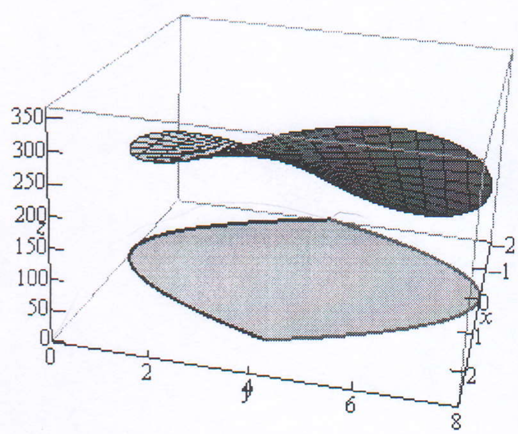
The volume of the solid that lies below the surface given by $z = f(x, y)$ and above the region D in the xy-plane is given by,

$$V = \iint_D f(x, y)\, dA$$

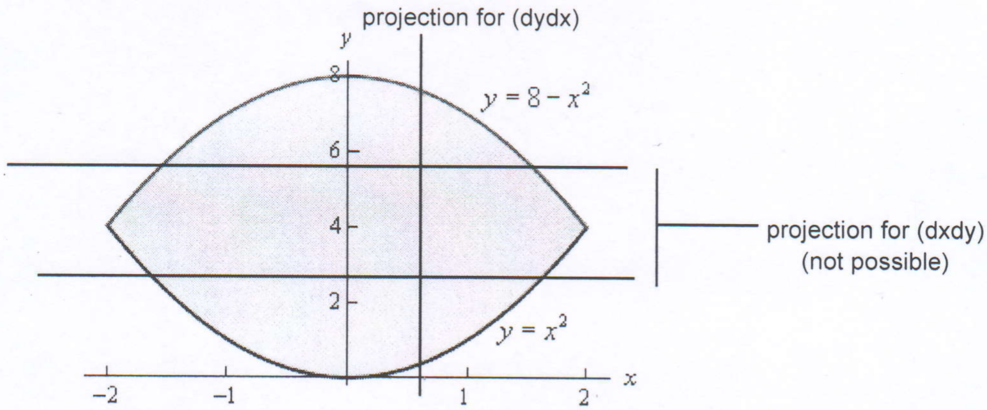
Example 3 Find the volume of the solid that lies below the surface given by $z = 16xy + 200$ and lies above the region in the xy-plane bounded by $y = x^2$ and $y = 8 - x^2$.

Solution
Here is the graph of the surface and we've tried to show the region in the xy-plane below the surface.





Here is a sketch of the region in the xy-plane by itself.



By setting the two bounding equations equal we can see that they will intersect at $x = 2$ and $x = -2$. So, the inequalities that will define the region D in the xy-plane are,

$$-2 \le x \le 2$$
$$x^2 \le y \le 8 - x^2$$

The volume is then given by,

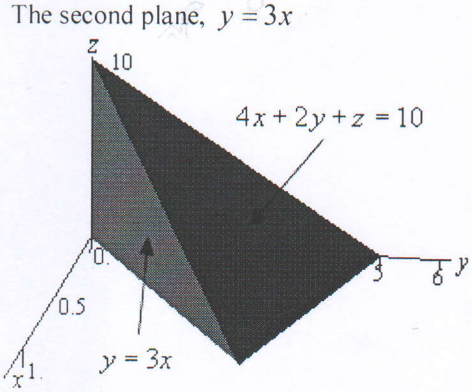
$$V = \iint_D 16xy + 200\, dA$$
$$= \int_{-2}^{2}\int_{x^2}^{8-x^2} 16xy + 200\, dy\, dx$$
$$= \int_{-2}^{2} \left(8xy^2 + 200y\right)\Big|_{x^2}^{8-x^2} dx$$
$$= \int_{-2}^{2} -128x^3 - 400x^2 + 512x + 1600\, dx$$
$$= \left(-32x^4 - \frac{400}{3}x^3 + 256x^2 + 1600x\right)\Bigg|_{-2}^{2} = \frac{12800}{3}$$

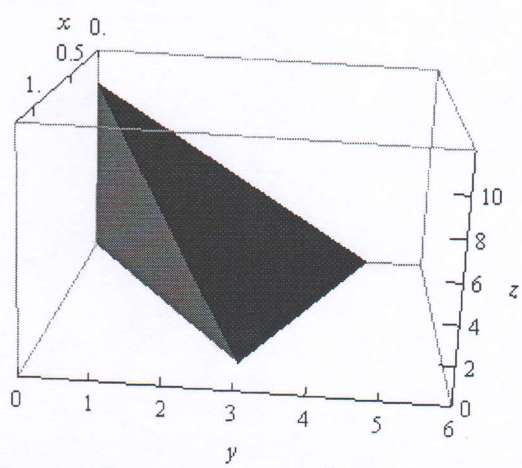
Example 4 Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, $x = 0$.

Solution

The first plane, $4x + 2y + z = 10$, **So** $z = 10 - 4x - 2y$

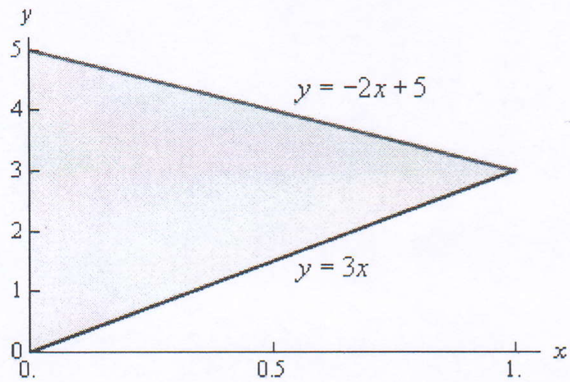
. The second plane, $y = 3x$





$0 + 4x + 2y = 10 \quad \Rightarrow \quad 2x + y = 5 \quad \Rightarrow \quad y = -2x + 5$

o, here is a sketch the region D.



$$0 \le x \le 1$$

$$3x \le y \le -2x + 5$$

$$V = \iint_D 10 - 4x - 2y\,dA$$

$$= \int_0^1 \int_{3x}^{-2x+5} 10 - 4x - 2y\,dy\,dx$$

$$= \int_0^1 \left(10y - 4xy - y^2\right)\Big|_{3x}^{-2x+5} dx$$

$$= \int_0^1 25x^2 - 50x + 25\,dx$$

$$= \left(\frac{25}{3}x^3 - 25x^2 + 25x\right)\Big|_0^1 = \frac{25}{3}$$