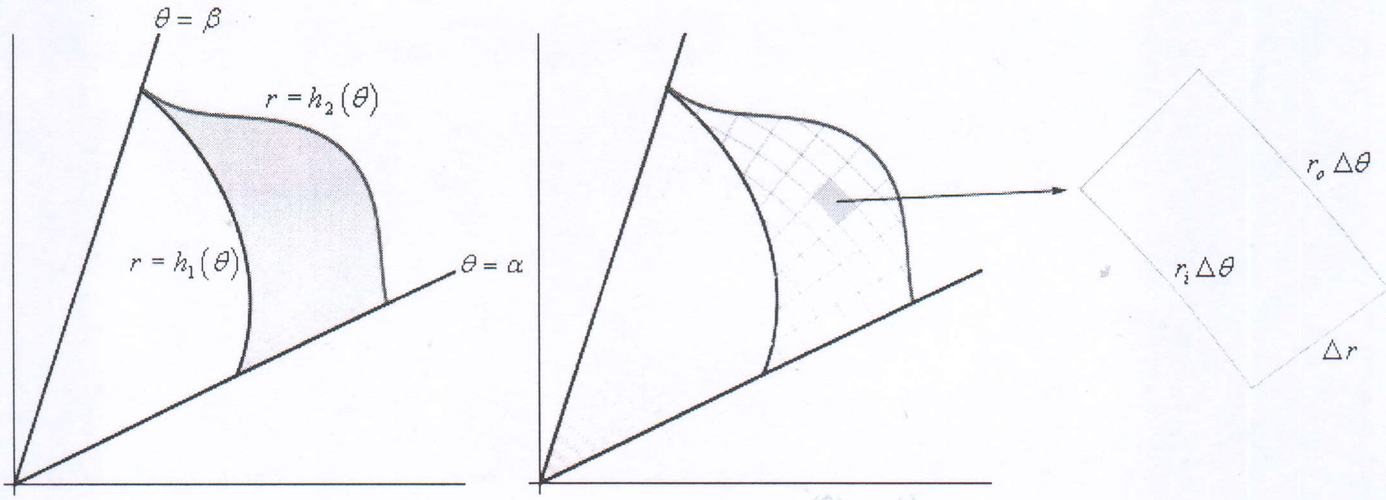
Double Integral in Polar Coordinates

a general region in terms of polar coordinates and see what we can do with that

. Here is a sketch of some region using polar coordinates.



So, our general region will be defined by inequalities,

$$\alpha \le \theta \le \beta$$

$$h_1(\theta) \le r \le h_2(\theta)$$

Now, to find dA let's redo the figure above as follows,

$$\boxed{dA = r\,dr\,d\theta}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

$$\boxed{\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Example 1 Evaluate the following integrals by converting them into polar coordinates.

(a) $\iint_D 2xy\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

(b) $\iint_D \mathbf{e}^{x^2+y^2}\,dA$, D is the unit circle centered at the origin.

Solution

(a) $\iint_D 2xy\,dA$, **D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.**

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between them so we will have the following inequality for r

26

$$2 \le r \le 5$$

Also since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2xy\,dA = \int_0^{\frac{\pi}{2}} \int_2^5 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

Don't forget to do the conversions and to add in the extra r. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2xy\,dA &= \int_0^{\frac{\pi}{2}} \int_2^5 r^3 \sin(2\theta)\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \sin(2\theta)\Big|_2^5\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4}\sin(2\theta)\,d\theta \\ &= -\frac{609}{8}\cos(2\theta)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{609}{4}\end{aligned}$$

(b) $\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA$, ***D*** **is the unit circle centered at the origin.**

In this case we can't do this integral in terms of Cartesian coordinates. We will however be able to do it in polar coordinates. First, the region D is defined by,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

In terms of polar coordinates the integral is then,

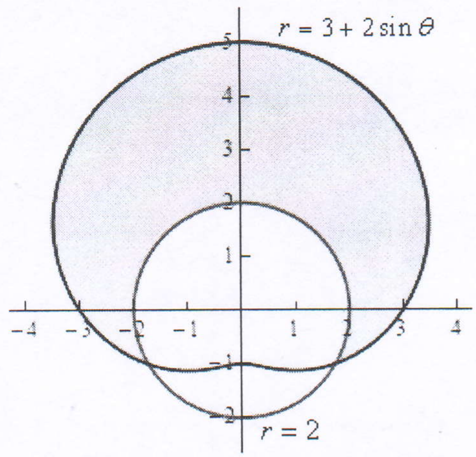
$$\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^1 r\,\mathbf{e}^{r^2}\,dr\,d\theta$$

Notice that the addition of the r gives us an integral that we can now do. Here is the work for this integral.

$$\begin{aligned}\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA &= \int_0^{2\pi} \int_0^1 r\,\mathbf{e}^{r^2}\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\mathbf{e}^{r^2}\Big|_0^1\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(\mathbf{e}-1)\,d\theta \\ &= \pi(\mathbf{e}-1)\end{aligned}$$

Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

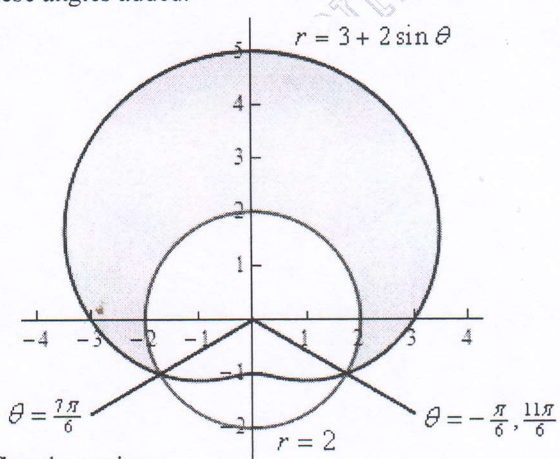
Solution
Here is a sketch of the region, *D*, that we want to determine the area of.



by setting the two equations and solving.

$$3 + 2\sin\theta = 2$$
$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$$
$$2 \le r \le 3 + 2\sin\theta$$

$$A = \iint_D dA$$
$$= \int_{-\pi/6}^{7\pi/6} \int_2^{3+2\sin\theta} r\,dr\,d\theta$$
$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}r^2 \Big|_2^{3+2\sin\theta} d\theta = \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\,d\theta$$
$$= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta)\,d\theta$$

$$= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta) \right) \Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187$$

Example 3 Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution

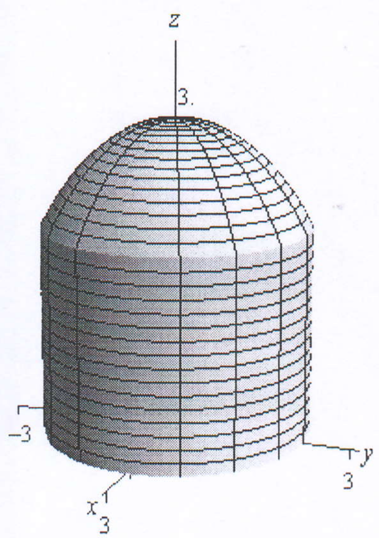
We know that the formula for finding the volume of a region is,

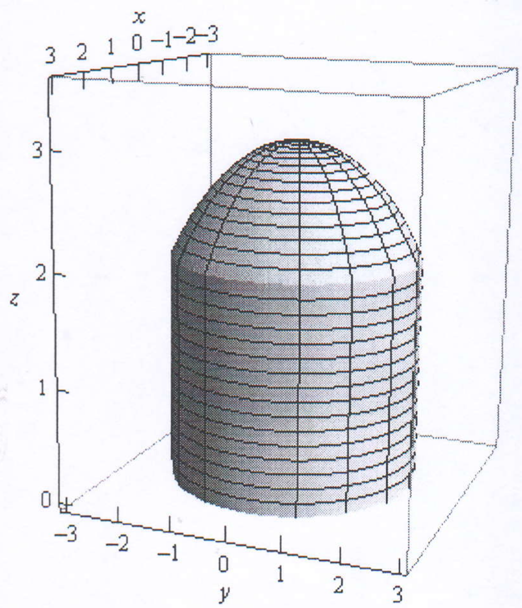
$$V = \iint_D f(x, y)\, dA$$

Here is the function.

$$z = \sqrt{9 - x^2 - y^2}$$

As we know that z= $f(x, y)$





$$0 \le \theta \le 2\pi$$

$$0 \le r \le \sqrt{5}$$

$$z = \sqrt{9 - (x^2 + y^2)} = \sqrt{9 - r^2}$$

$$V = \iint_D \sqrt{9 - x^2 - y^2}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}(9 - r^2)^{\frac{3}{2}} \Bigg|_0^{\sqrt{5}} d\theta$$

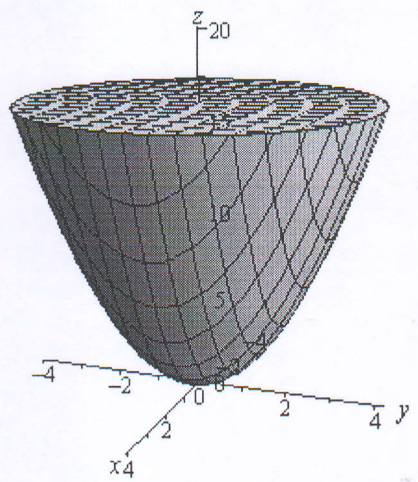
$$= \int_0^{2\pi} \frac{19}{3} d\theta$$

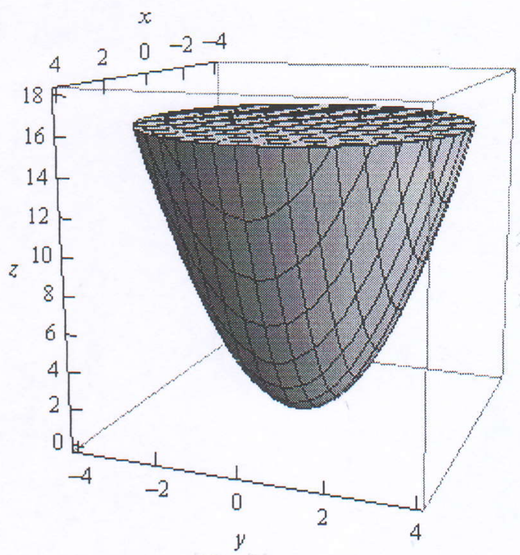
$$= \frac{38\pi}{3}$$

Example 4 Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution

Let's start this example off with a quick sketch of the region.





$$V = \iint_D f(x,y)\,dA = \iint_D \{\, f(x,y)\Big|_{\text{upper}} - f(x,y)\Big|_{\text{Lower}} \,\}\, dA$$

so

$$V = \iint_D \{(16) - (x^2 + y^2)\}\, dA$$

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 4 \qquad z = 16 - r^2$$

me is then,

$$\begin{aligned} V &= \iint_D 16 - (x^2 + y^2)\, dA \\ &= \int_0^{2\pi} \int_0^4 r(16 - r^2)\, dr\, d\theta \\ &= \int_0^{2\pi} \left(8r^2 - \frac{1}{4} r^4\right)\Bigg|_0^4 d\theta \\ &= \int_0^{2\pi} 64\, d\theta \\ &= 128\pi \end{aligned}$$

Example 5 Evaluate the following integral by first converting to polar coordinates.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2)\, dx\, dy$$

Solution

$$0 \le y \le 1$$

$$0 \le x \le \sqrt{1 - y^2}$$

$$x = \sqrt{1 - y^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

$$dx\,dy = dA = r\,dr\,d\theta$$

and so the integral becomes,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right) dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\left(r^2\right) dr\,d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right) dx\,dy &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\left(r^2\right)\Big|_0^1 \,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(1)\,d\theta \\ &= \frac{\pi}{4}\sin(1)\end{aligned}$$