Triple Integrals

The notation for the general triple integrals is,

$$\iiint_E f(x,y,z)\,dV$$

Let's start simple by integrating over the box,

$$B = [a,b]\times[c,d]\times[r,s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is,

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example 1 Evaluate the following integral.

$$\iiint_B 8xyz\,dV\,, \quad B = [2,3]\times[1,2]\times[0,1]$$

Solution

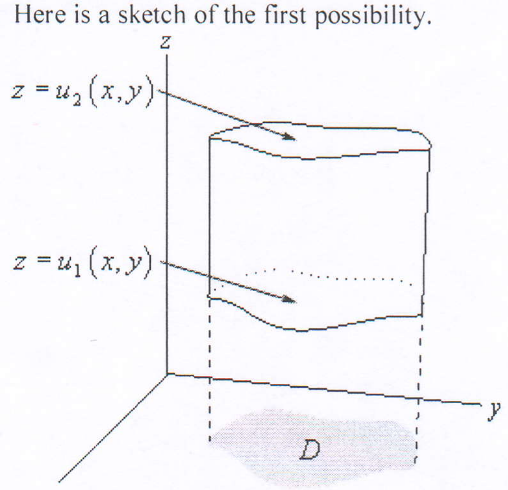
Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}\iiint_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\ &= \int_1^2 \int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\ &= \int_1^2 \int_2^3 4xy\,dx\,dy \\ &= \int_1^2 2x^2y\Big|_2^3\,dy \\ &= \int_1^2 10y\,dy = 15\end{aligned}$$

Fact

The volume of the three-dimensional region E is given by the integral, $V = \iiint_E dV$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.

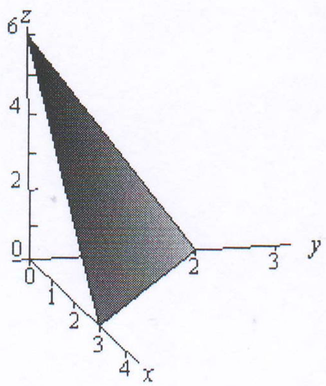


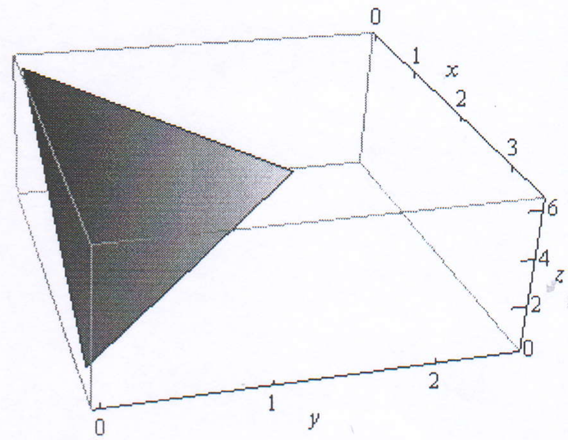
In this case we will evaluate the triple integral as follows,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

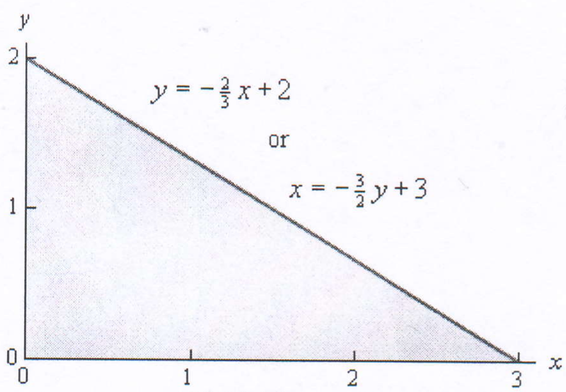
Example 2 Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.

Solution





. So D will be the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,2)$. Here is a sketch of D.



$$0 \le z \le 6-2x-3y$$

We can integrate the double integral over D using either of the following two sets of inequalit

$0 \le x \le 3$ $0 \le y \le -\frac{2}{3}x+2$	$0 \le x \le -\frac{3}{2}y+3$ $0 \le y \le 2$

$$\iiint_E 2x\,dV = \iint_D \left[\int_0^{6-2x-3y} 2x\,dz \right] dA$$

$$= \iint_D 2xz\Big|_0^{6-2x-3y}\,dA$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x(6-2x-3y)\,dy\,dx$$

$$= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2} dx$$

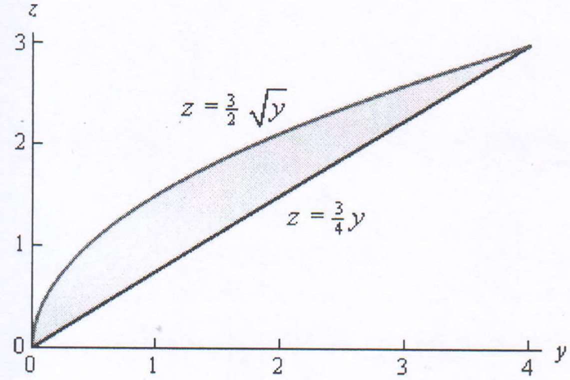
$$= \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx$$

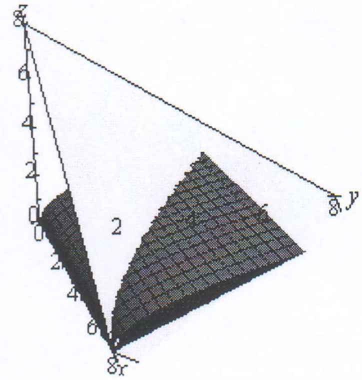
$$= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Bigg|_0^3$$

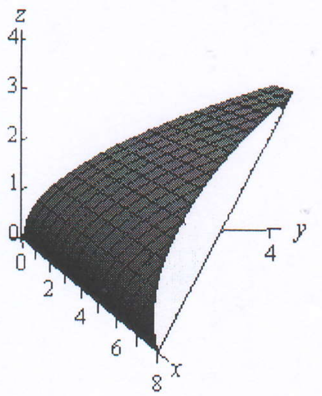
$$=9$$

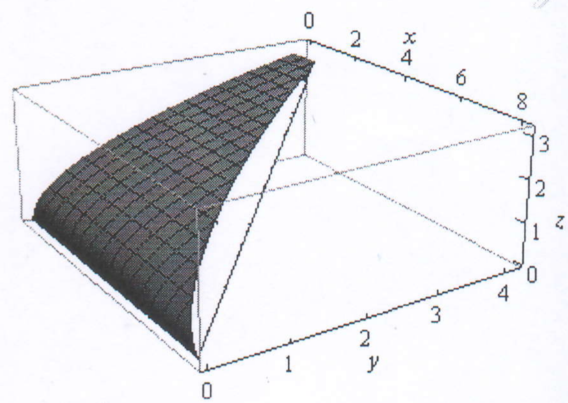
Example 3 Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

Solution









Here are the limits for each of the variables.

$$0 \le y \le 4$$

$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8 - y - z$$

$$V = \iiint_E dV = \iint_D \left[\int_0^{8-y-z} dx \right] dA$$

$$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy$$

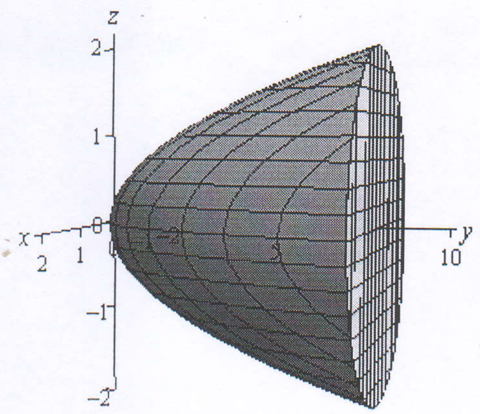
$$= \int_0^4 \left(8z - yz - \frac{1}{2}z^2 \right)\Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy$$

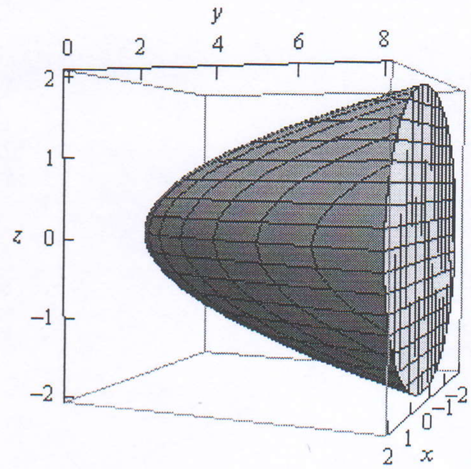
$$= \int_0^4 12y^{\frac{1}{2}} - \frac{57}{8}y - \frac{3}{2}y^{\frac{3}{2}} + \frac{33}{32}y^2 \, dy$$

$$= \left(8y^{\frac{3}{2}} - \frac{57}{16}y^2 - \frac{3}{5}y^{\frac{5}{2}} + \frac{11}{32}y^3 \right)\Bigg|_0^4 = \frac{49}{5}$$

Example 4 Evaluate $\iiint_E \sqrt{3x^2+3z^2}\,dV$ where E is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

Solution Here is a sketch of the solid E.





$$2x^2 + 2z^2 = 8 \qquad \Rightarrow \qquad x^2 + z^2 = 4$$

$$x = r\cos\theta \qquad\qquad z = r\sin\theta$$

$$x^2 + z^2 = r^2$$

$$2x^2 + 2z^2 \le y \le 8$$

$$0 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$$

$$= \iint_D \left(y\sqrt{3x^2+3z^2} \right)\Big|_{2x^2+2z^2}^{8} dA$$

$$= \iint_D \sqrt{3\left(x^2+z^2\right)}\left(8 - \left(2x^2+2z^2\right)\right) dA$$

$$\sqrt{3\left(x^2+z^2\right)}\left(8 - \left(2x^2+2z^2\right)\right) = \sqrt{3r^2}\left(8-2r^2\right)$$

$$= \sqrt{3}\, r\left(8-2r^2\right)$$

$$= \sqrt{3}\left(8r - 2r^3\right)$$

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \sqrt{3}\,\left(8r-2r^3\right) dA$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(8r - 2r^3\right) r\,dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi} \left(\frac{8}{3}r^3 - \frac{2}{5}r^5\right)\Big|_0^2 d\theta$$

$$= \sqrt{3}\int_0^{2\pi} \frac{128}{15}\,d\theta = \frac{256\sqrt{3}\,\pi}{15}$$