Triple Integrals in Cylindrical Coordinates

The following are the conversion formulas for cylindrical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$dV = r\,dz\,dr\,d\theta$$

In terms of cylindrical coordinates a triple integral is,

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Example 1 Evaluate $\iiint_E y\,dV$ where E is the region that lies below the plane $z = x + 2$ above the *xy*-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

$$0 \le z \le x + 2 \qquad \Rightarrow \qquad 0 \le z \le r\cos\theta + 2$$

Next, the region D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the *xy*-plane and so the ranges for it are,

$$0 \le \theta \le 2\pi \qquad 1 \le r \le 2$$

Remember that we are above the *xy*-plane and so we are above the plane $z = 0$

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} (r\sin\theta)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta + 2)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin(2\theta) + 2r^2\sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{8}r^4\sin(2\theta) + \frac{2}{3}r^3\sin\theta\right)\Bigg|_1^2 d\theta \\
&= \int_0^{2\pi} \frac{15}{8}\sin(2\theta) + \frac{14}{3}\sin\theta\,d\theta \\
&= \left(-\frac{15}{16}\cos(2\theta) - \frac{14}{3}\cos\theta\right)\Bigg|_0^{2\pi} \\
&= 0
\end{aligned}$$

Example 2 Convert $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

Solution

Here are the ranges of the variables from this iterated integral.

$$-1 \le y \le 1$$

$$0 \le x \le \sqrt{1-y^2}$$

$$x^2 + y^2 \le z \le \sqrt{x^2 + y^2}$$

from Integral Limits $\qquad x = \sqrt{1-y^2}$ and $x = 0$ $\qquad$ equalize the limit of x

$$-1 \le y \le 1 \quad \text{Limits of y}$$

then $$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 1$$

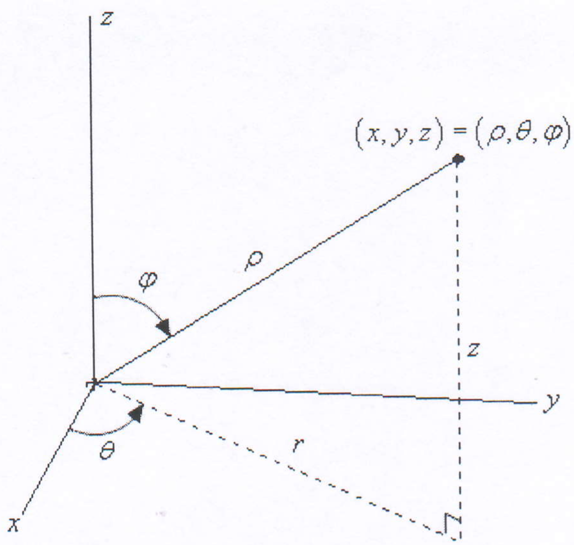
$$r^2 \le z \le r$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r\left(r\cos\theta\right)\left(r\sin\theta\right)z\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta$$

Triple Integrals in Spherical coordinates

The following sketch shows the relationship between the Cartesian and spherical coordinate systems.



Here are the conversion formulas for spherical coordinates.

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

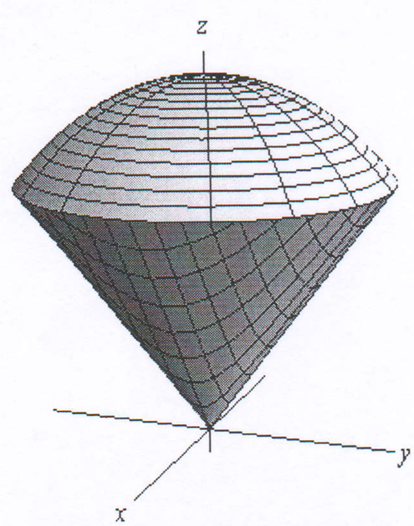
We also have the following restrictions on the coordinates.

$$\rho \geq 0 \qquad 0 \leq \varphi \leq \pi$$

For our integrals we are going to restrict E down to a spherical wedge. This will mean that we are going to take ranges for the variables as follows,

$$a \leq \rho \leq b \qquad \alpha \leq \theta \leq \beta \qquad \delta \leq \varphi \leq \gamma$$

Here is a quick sketch of a spherical wedge in which the lower limit for both ρ and φ are zero for reference purposes. Most of the wedges we'll be working with will fit into this pattern.



also

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Therefore the integral will become,

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_\alpha^\beta \int_\delta^\gamma \rho^2 \sin\varphi \; f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,d\rho\,d\theta\,d\varphi$$

Example 1 Evaluate $\iiint_E 16z\,dV$ where E is the upper half of the sphere $x^2+y^2+z^2=1$.

Solution

Since we are taking the upper half of the sphere the limits for the variables are,

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

The integral is then,

$$\iiint_E 16z\,dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi (16\rho\cos\varphi)\,d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 8\rho^3 \sin(2\varphi)\,d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin(2\varphi)\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}} 4\pi\sin(2\varphi)\,d\varphi$$
$$= -2\pi\cos(2\varphi)\Big|_0^{\frac{\pi}{2}}$$
$$= 4\pi$$

Example 2 Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy$ into spherical coordinates.

Solution

Let's first write down the limits for the variables.

$$0 \le y \le 3$$
$$0 \le x \le \sqrt{9-y^2}$$
$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

(since this is the angle around the z-axis).

$$0 \le \theta \le \frac{\pi}{2}$$

The lower bound, $z=\sqrt{x^2+y^2}$, The upper bound, $z=\sqrt{18-x^2-y^2}$

upper half of the sphere, $x^2+y^2+z^2=18$ and so from this we now have the following range for ρ

$$0 \le \rho \le \sqrt{18} = 3\sqrt{2}$$

Now all that we need is the range for φ. There are two ways to get this. One is from where the cone and the sphere intersect. Plugging in the equation for the cone into the sphere gives,

$$\left(\sqrt{x^2+y^2}\right)^2 + z^2 = 18$$
$$z^2+z^2=18$$
$$z^2=9$$
$$z=3$$

we know that $\rho = 3\sqrt{2}$ since we are intersecting on the sphere. This gives,

$$\rho\cos\varphi = 3$$

$$3\sqrt{2}\cos\varphi = 3$$

$$\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

So, it looks like we have the following range,

$$0 \le \varphi \le \frac{\pi}{4}$$

The other way to get this range is from the cone by itself. By first converting the equation into cylindrical coordinates and then into spherical coordinates we get the following,

$$z = r$$

$$\rho\cos\varphi = \rho\sin\varphi$$

$$1 = \tan\varphi \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

So, recalling that $\rho^2 = x^2 + y^2 + z^2$, the integral is then,

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz\, dx\, dy = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin\varphi \, d\rho\, d\theta\, d\varphi$$