Change of Variables

substitution rule that told us that,

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_c^d f(u)\,du \qquad \text{where} \quad u = g(x)$$

First we need a little notation out of the way. We call the equations that define the change of variables a **transformation**. Also we will typically start out with a region, R, in xy-coordinates and transform it into a region in uv-coordinates.

Example 1 Determine the new region that we get by applying the given transformation to the region R.

(a) R is the ellipse $x^2 + \dfrac{y^2}{36} = 1$ and the transformation is $x = \dfrac{u}{2}$, $y = 3v$.

(b) R is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \dfrac{x}{3} - \dfrac{4}{3}$ and the transformation is $x = \dfrac{1}{2}(u + v)$, $y = \dfrac{1}{2}(u - v)$.

Solution

(a) R is the ellipse $x^2 + \dfrac{y^2}{36} = 1$ and the transformation is $x = \dfrac{u}{2}$, $y = 3v$.

There really isn't too much to do with this one other than to plug the transformation into the equation for the ellipse and see what we get.

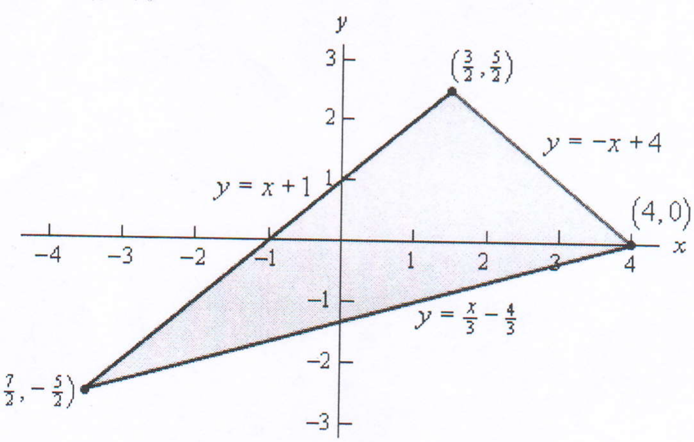
$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

So, we started out with an ellipse and after the transformation we had a disk of radius 2.

(b) R is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \dfrac{x}{3} - \dfrac{4}{3}$ and the transformation is $x = \dfrac{1}{2}(u + v)$, $y = \dfrac{1}{2}(u - v)$.



Let's do $y = -x + 4$ first. Plugging in the transformation gives

$$\frac{1}{2}(u - v) = -\frac{1}{2}(u + v) + 4$$

$$u - v = -u - v + 8$$

$$2u = 8$$

$$u = 4$$

Now let's take a look at $y = x + 1$,

$$\frac{1}{2}(u - v) = \frac{1}{2}(u + v) + 1$$

$$u - v = u + v = 2$$

$$-2v = 2$$

$$v = -1$$

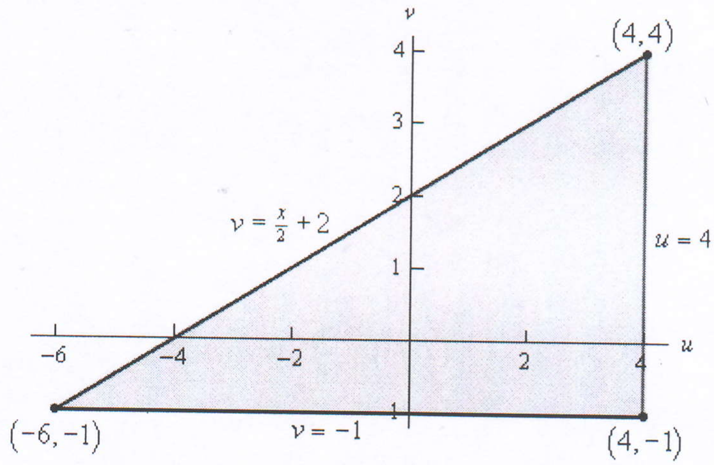
Finally, let's transform $y = \frac{x}{3} - \frac{4}{3}$.

$$\frac{1}{2}(u - v) = \frac{1}{3}\left(\frac{1}{2}(u + v)\right) - \frac{4}{3}$$

$$3u - 3v = u + v - 8$$

$$4v = 2u + 8$$

$$v = \frac{u}{2} + 2$$



Definition

The **Jacobian** of the transformation $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian is defined as a determinant of a 2x2 matrix, if you are unfamiliar with this that is okay. Here is how to compute the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore, another formula for the determinant is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variables for a Double Integral

Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x = g(u,v)$, $y = h(u,v)$ the region becomes S and the integral becomes,

$$\iint_D f(x,y)\,dA = \iint_S f\big(g(u,v), h(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

If we look just at the differentials in the above formula we can also say that

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Example 2 Show that when changing to polar coordinates we have $dA = r\,dr\,d\theta$

Solution The transformation here is the standard conversion formulas,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

The Jacobian for this transformation is,

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}\\ &= \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}\\ &= r\cos^2\theta - \left(-r\sin^2\theta\right)\\ &= r\left(\cos^2\theta + \sin^2\theta\right)\\ &= r\end{aligned}$$

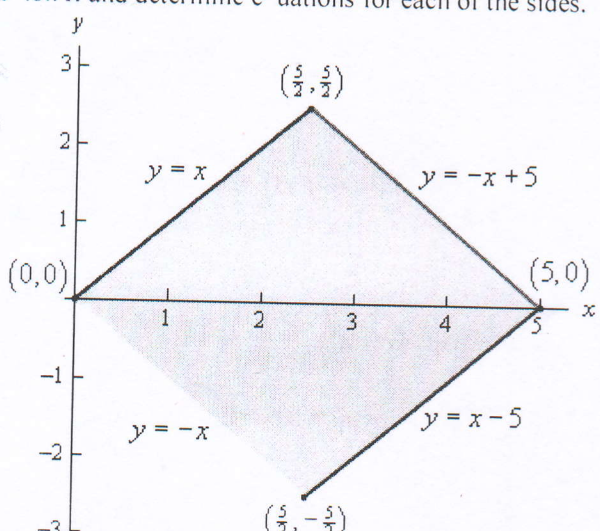
We then get,

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta$$

Example 3 Evaluate $\iint_R x + y\,dA$ where R is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $\left(\frac{5}{2},\frac{5}{2}\right)$ and $\left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

Solution

First, let's sketch the re ion R and determine e uations for each of the sides.



Let's use the transformation and see what we get. We'll do this by plugging the transformation into each of the equations above.

Let's start the process off with $y = x$.

$$\begin{aligned} 2u - 3v &= 2u + 3v \\ 6v &= 0 \\ v &= 0 \end{aligned}$$

Transforming $y = -x$ is similar.

$$\begin{aligned} 2u - 3v &= -(2u + 3v) \\ 4u &= 0 \\ u &= 0 \end{aligned}$$

Next we'll transform $y = -x + 5$.

$$\begin{aligned} 2u - 3v &= -(2u + 3v) + 5 \\ 4u &= 5 \\ u &= \frac{5}{4} \end{aligned}$$

Finally, let's transform $y = x - 5$.

$$\begin{aligned} 2u - 3v &= 2u + 3v - 5 \\ -6v &= -5 \\ v &= \frac{5}{6} \end{aligned}$$

The region S is then a rectangle whose sides are given by $u = 0$, $v = 0$, $u = \frac{5}{4}$ and $v = \frac{5}{6}$ and s the ranges of u and v are,

$$0 \le u \le \frac{5}{4} \qquad\qquad 0 \le v \le \frac{5}{6}$$

Next, we need the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6 - 6 = -12$$

The integral is then,

$$\begin{aligned} \iint\limits_R x + y\,dA &= \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} (2u + 3v) + (2u - 3v)\left|-12\right| du\,dv \\ &= \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} 48u\,du\,dv \\ &= \int_0^{\frac{5}{6}} 24u^2 \Big|_0^{\frac{5}{4}} dv \\ &= \int_0^{\frac{5}{6}} \frac{75}{2}\,dv \\ &= \frac{75}{2} v \Big|_0^{\frac{5}{6}} \\ &= \frac{125}{4} \end{aligned}$$

Example 4 Evaluate $\iint\limits_R x^2 - xy + y^2\, dA$ where R is the ellipse given by $x^2 - xy + y^2 = 2$ and using the transformation $x = \sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v$, $y = \sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v$.

Solution

The first thing to do is to plug the transformation into the equation for the ellipse to see what the region transforms into.

$$
\begin{aligned}
2 &= x^2 - xy + y^2 \\
&= \left(\sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v\right)^2 - \left(\sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v\right)\left(\sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v\right) + \left(\sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v\right)^2 \\
&= 2u^2 - \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2 - \left(2u^2 - \frac{2}{3}v^2\right) + 2u^2 + \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2 \\
&= 2u^2 + 2v^2
\end{aligned}
$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

The integral is then,

$$\iint\limits_R x^2 - xy + y^2\, dA = \iint\limits_S 2\left(u^2 + v^2\right)\left|\frac{4}{\sqrt{3}}\right| du\, dv$$

Do not make the mistake of substituting $x^2 - x \quad + \quad {}^2 = 2$ or $u^2 + v^2 = 1$ in for the integrands. the integral out will convert to polar coordinates.

$$
\begin{aligned}
\iint\limits_R x^2 - xy + y^2\, dA &= \iint\limits_S 2\left(u^2 + v^2\right)\left|\frac{4}{\sqrt{3}}\right| du\, dv \\
&= \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 \left(r^2\right) r\, dr\, d\theta \\
&= \frac{8}{\sqrt{3}} \int_0^{2\pi} \left.\frac{1}{4} r^4 \right|_0^1 d\theta \\
&= \frac{8}{\sqrt{3}} \int_0^{2\pi} \frac{1}{4} d\theta \\
&= \frac{4\pi}{\sqrt{3}}
\end{aligned}
$$

Let's now briefly look at triple integrals. In this case we will again start with a region R and use the transformation $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$ to transform the region into the new region S. To do the integral we will need a Jacobian, just as we did with double integrals. Here is the definition of the Jacobian for this kind of transformation.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The integral under this transformation is,

$$\iiint_R f(x,y,z)\,dV = \iiint_S f\big(g(u,v,w),h(u,v,w),k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

As with double integrals we can look at just the differentials and note that we must have

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Example 5 Verify that $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$ when using spherical coordinates.

Solution
Here the transformation is just the standard conversion formulas.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

The Jacobian is,

$$\begin{aligned}\frac{\partial(z,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\ &= -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin\varphi\cos^2\varphi\sin^2\theta + 0 \\ &\qquad -\rho^2\sin^3\varphi\sin^2\theta - 0 - \rho^2\sin\varphi\cos^2\varphi\cos^2\theta \\ &= -\rho^2\sin^3\varphi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\ &= -\rho^2\sin^3\varphi - \rho^2\sin\varphi\cos^2\varphi \\ &= -\rho^2\sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\ &= -\rho^2\sin\varphi\end{aligned}$$

Finally, dV becomes,

$$dV = \left|-\rho^2\sin\varphi\right| d\rho\, d\theta\, d\varphi = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

Recall that we restricted φ to the range $0 \le \varphi \le \pi$ for spherical coordinates and so we know that $\sin\varphi \ge 0$ and so we don't need the absolute value bars on the sine.

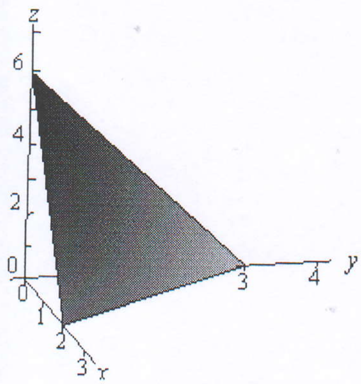
Surface Area

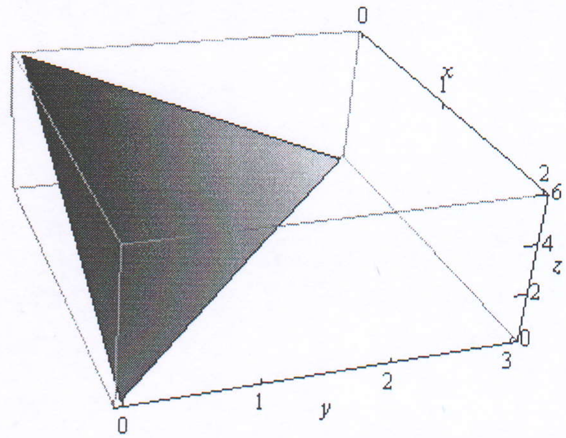
Here we want to find the surface area of the surface given by $z = f(x,y)$ where (x,y) is a point from the region D in the xy-plane. In this case the surface area is given by,

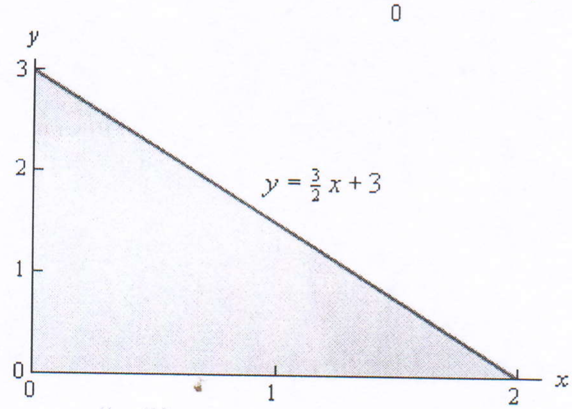
$$S = \iint\limits_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\, dA$$

Example 1 Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution







$z = f(x,y)$ then $z = 6 - 3x - 2y$ so $f_x = -3$ and $f_y = -2$

The limits defining D are,

$$0 \le x \le 2 \qquad 0 \le y \le -\frac{3}{2}x + 3$$

The surface area is then,

$$S = \iint\limits_D \sqrt{[-3]^2 + [-2]^2 + 1}\, dA$$

$$= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14}\, dy\, dx$$

$$= \sqrt{14} \int_0^2 -\frac{3}{2}x + 3\, dx$$

$$= \sqrt{14}\left(-\frac{3}{4}x^2 + 3x \right)\Bigg|_0^2$$

$$= 3\sqrt{14}$$

Example 2 Determine the surface area of the part of $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

Solution Here are the partial derivatives,

$$f_x = y \qquad f_y = x$$

The integral for the surface area is,

$$S = \iint_D \sqrt{x^2 + y^2 + 1}\, dA$$

Given that D is a disk it makes sense to do this integral in polar coordinates.

$$\begin{aligned} S &= \iint_D \sqrt{x^2 + y^2 + 1}\, dA \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{1 + r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\frac{2}{3}\right)\left(1 + r^2\right)^{\frac{3}{2}}\Big|_0^1\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right) d\theta \\ &= \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right) \end{aligned}$$

Area and Volume Revisited

We'll first look at the area of a region. The area of the region D is given by,

$$\text{Area of } D = \iint\limits_D dA$$

Now let's give the two volume formulas. First the volume of the region E is given by,

$$\text{Volume of } E = \iiint\limits_E dV$$

Finally, if the region E can be defined as the region under the function $z = f\left(x, y\right)$ and above the region D in xy-plane then,

$$\text{Volume of } E = \iint\limits_D f\left(x, y\right)\, dA$$

Note as well that there are similar formulas for the other planes. For instance, the volume of the region behind the function $y = f\left(x, z\right)$ and in front of the region D in the xz-plane is given by,

$$\text{Volume of } E = \iint\limits_D f\left(x, z\right)\, dA$$

Likewise, the the volume of the region behind the function $x = f\left(y, z\right)$ and in front of the region D in the yz-plane is given by,

$$\text{Volume of } E = \iint\limits_D f\left(y, z\right)\, dA$$

Home Work:

1 Evaluate:

(a) $\int_1^3 \int_0^2 (y^3 - xy)\, dy\, dx$

(b) $\int_0^a dx \int_0^{y_1} (x - y)\, dy$, where $y_1 = \sqrt{a^2 - x^2}$

2 Determine:

(a) $\int_0^{\sqrt{3}+2} \int_0^{\pi/3} (2\cos\theta - 3\sin 3\theta)\, d\theta\, dr$

(b) $\int_2^4 \int_1^2 \int_0^4 xy(z+2)\, dx\, dy\, dz$

(c) $\int_0^1 dz \int_1^2 dx \int_0^x (x + y + z)\, dy$

3 Form a double integral to represent the area of the plane figure bounded by the polar curve $r = 3 + 2\cos\theta$ and the radius vectors at $\theta = 0$ and $\theta = \pi/2$, and evaluate it.

(4) Use iterated integration to compute the double integral of the rectangular re $\iint_R x^2 y \, dA$ over the rectang $R = \{(x, y) \mid 1 \le x \le 2, 0 \le y \le 1\}$.

(5) Use iterated integration to compute the double integral of the rectangular r $\iint_R 2x e^y \, dA$ over the rectang $R = \{(x, y) \mid -1 \le x \le 0, 0 \le y \le \ln(2)\}$.

(6)Use iterated integration to compute the double integral of the rectangular regi $\iint_R \sin(x + y)\, dA$ over the rectang $R = \left\{(x, y) \middle| 0 \le x \le \frac{\pi}{4}, 0 \le y \le \frac{\pi}{2}\right\}$.

(7) Use iterated integration to compute the double integral of the rectangular re $\iint_R x \sin(xy)\, dA$ over the rectang $R = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le 1\}$.

(8) Find the volume of the solid bounded below by the rectan $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2\}$ in the xy-plane andabov the graph of $z = f(x, y) = 2x + 3y$.

(9) Find the volume of the solid bounded below by the rectan $R = \{(x, y) \mid 1 \le x \le 2, 1 \le y \le e\}$ in the xy-plane and abov the graph of $z = f(x, y) = x \ln(xy)$.

Further problems

1 Evaluate $\int_0^{\pi}\int_0^{\cos\theta} r\sin\theta \,\mathrm{d}r\,\mathrm{d}\theta$

2 Evaluate $\int_0^{2\pi}\int_0^{3} r^3(9-r^2)\,\mathrm{d}r\,\mathrm{d}\theta$

3 Evaluate $\int_0^2 \mathrm{d}x \int_1^3 \mathrm{d}y \int_1^2 xy^2z\,\mathrm{d}z$

4 Evaluate $\int_0^2 \mathrm{d}x \int_1^2 (x^2+y^2)\,\mathrm{d}y$

5 The base of a solid is the plane figure in the x–y plane bounded by $x=0$, $x=2$, $y=x$ and $y=x^2+1$. The sides are vertical and the top is the surface $z=x^2+y^2$. Calculate the volume of the solid so formed.

6 A solid consists of vertical sides standing on the plane figure enclosed by

7 Evaluate the iterated integral $\int_0^2\int_0^1 (x^2+xy+y^2)\,dy\,dx$.

8 Evaluate the iterated integral $\int_1^2\int_0^{\pi} x\cos y\,dy\,dx$.

9 Evaluate the iterated integral $\int_0^{\ln(2)}\int_0^1 e^{x+2y}\,dx\,dy$.

10 Find the volume enclosed by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=5-x$.

11 Determine the volume of the solid bounded by the surfaces $y=x^2$, $x=y^2$, $z=2$ and $x+y+z=4$.

12 Find the volume of the solid bounded by the plane $z=0$, the cylinder $x^2+y^2=a^2$ and the surface $z=x^2+y^2$.

13 Find the volume of the solid in the first octant bounded by the planes $x=0$, $y=0$, $z=0$, $z=x+y$ and the surface $x^2+y^2=a^2$.

14 Calculate $\iint x^2y^2\,\mathrm{d}x\,\mathrm{d}y$ over the triangular region in the x–y plane with vertices $(0, 0)$, $(1, 1)$, $(1, 2)$.