Lecture Thirteen
Linear and Angular Momentum

1- *Dynamic Forces on Curve Surfaces due to the Impingement Liquid Jets.*

In determination of the force and energy transfer between the moving blades and the fluid, the relative velocity between the blade and the fluid becomes very important effective factor in calculations. The following parameters and notations will be used in our calculation of dynamic forces.

α_1 , angle with direction of motion of the vane, at which the jet enters the vane.

α_2 , angle with direction of motion at which the jet leaves the vane.

θ_1 & θ_2 , angles which V_{r1} and V_{r2} makes with direction of motion of vane.

V_1 & V_2 absolute velocities of jet at inlet & leaving the vane.

V_{r1} & V_{r2} relative velocity at entrance and exit the vane, $V_r = V - u$

V_{w1}&V_{w2}, horizontal components of V_1 & V_2 respectively.

V_{f1}&V_{f2}, vertical components of V_1 & V_2 respectively.

F_c is the force applied on the ***C.V*** by the vane therefore from Eq.6 in L-12 the momentum theorem in x-direction as,

$$F_c = \rho Q[(V_{rx})_{out} - (V_{rx})_{in}] = \dot{m}[V_{r2}\cos\theta_2 - V_{r1}\cos\theta_1] \quad (1)$$

Let the force R_x has to be act opposite to F_c

$$R_x = -F_c = \dot{m}[V_{r1}\cos\theta_1 - V_{r2}\cos\theta_2] \quad (2)$$

Power developed by the vane is given by

$$P = uR_x = u * \dot{m}[V_{r1}\cos\theta_1 - V_{r2}\cos\theta_2] \quad (3)$$

From the outlet velocity triangle, it can be written

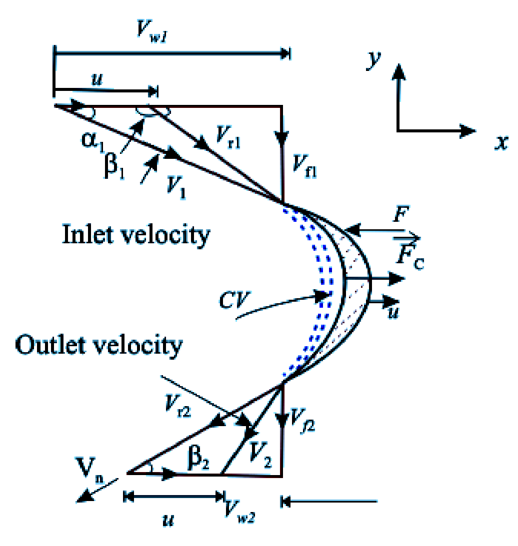


Figure 1: Flow of fluid along a moving curved plane

$(V_{w2}+u)^2 = V_{r2}^2 - V_{f2}^2$

Or $V_{w2}^2 + u^2 + 2V_{w2}u = V_{r2}^2 - V_{f2}^2$

Or $V_2^2 - V_{f2}^2 + u^2 + 2V_{w2}u = V_{r2}^2 - V_{f2}^2$

Or $V_{w2}u = \frac{1}{2}(V_{r2}^2 - V_2^2 - u^2)$ (4)

Similarly from the inlet velocity triangle. It is possible to write

$V_{w1}u = \frac{1}{2}(-V_{r1}^2 + V_1^2 + u^2)$ (5)

Addition of Eq's. (4 and 5) with no losses in relative velocity gives

$(V_{w1}+V_{w2})u = \frac{1}{2}(V_1^2 - V_2^2)$ (6)

Power of jet $= \dot{m}V_1^2/2$ (7)

The efficiency of the vane in developing power is given by

$\eta = \frac{out\ power}{input\ power} = \frac{uR_x}{\frac{\dot{m}}{2}V_1^2}$ (8)

Ex.1

A jet of water moving at 60 m/s is deflected by a vane moving at 25 m/s in a direction at 30° to the direction of the jet. The water leaves the blade normally to the motion of the vanes. Draw inlet and outlet triangle of velocities and find the vane angles for no shock at entering & exit. Take relative velocity at outlet equal to ($0.85V_{r1}$) and calculate the force on the vane of the jet diameter equal to (10) cm

Sol.

$u = 25\frac{m}{s}$; $V_1 = 60\frac{m}{s}$; $\propto_1 = 30°$

From triangle (ADC) as in below figure

$V_{w1} = 60\cos 30$

$V_{w1} = 60 * 0.866 = 51.96\frac{m}{s}$

$V_{f1} = V_1 \sin 30$

$V_{f1} = 60 * 0.5 = 30\ m/s$

$tan\theta_1 = \frac{CD}{AD-AB} = \frac{V_{f1}}{V_{w1}-U}$

$tan\theta_1 = \frac{30}{51.96-25} = 1.1127$

$\theta_1 = 48°4'$

$V_{r1} = \frac{V_{f1}}{sin\theta_1} = \frac{30}{0.7437} = 40.34\frac{m}{s}$

From triangle EFG

$V_{r2} = 0.85\,V_{r1} = 0.85 * 40.34 = 34.29\frac{m}{s}$

$cos\theta_2 = \frac{FG}{FE} = \frac{u}{V_{r2}} = \frac{25}{34.29} = 0.729$

$\theta_2 = 43°12'$

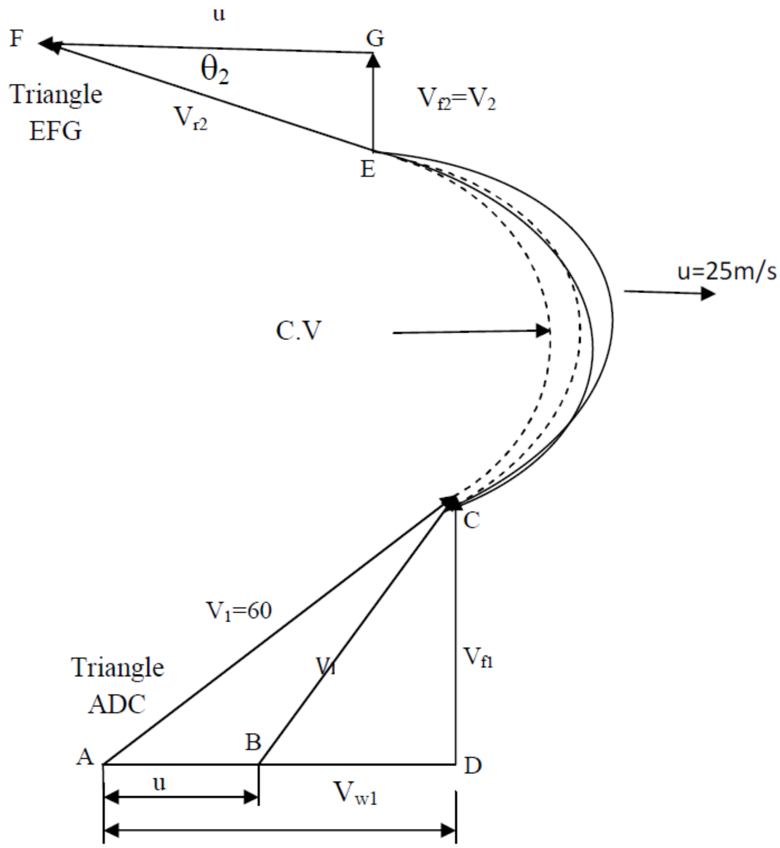
$\dot{m}_{r1} = \rho A V_{r1} = 1000 * \frac{\pi(0.1)^2}{4} * 40.34 = 316.7\ kg/s$

$\dot{m}_{r2} = \rho A V_{r2} = m_{r1} * 0.85 = 269.1\ kg/s$

$F_c = \dot{m}(V_{r2}cos\theta_2 - V_{r1}cos\theta_1)$

$F_c = 269.9 * 34.29\cos 43° - 316.7 * 40.34 * cos48°4'$

$F_c = -1800\ N$
$R = -F_c = 1800\ N$



2- *Jet engine.*

A turbo jet engine as shown in Fig. 2 is consists essentially of

- a compressor
- a combustion chamber
- a gas turbine
- a nozzle

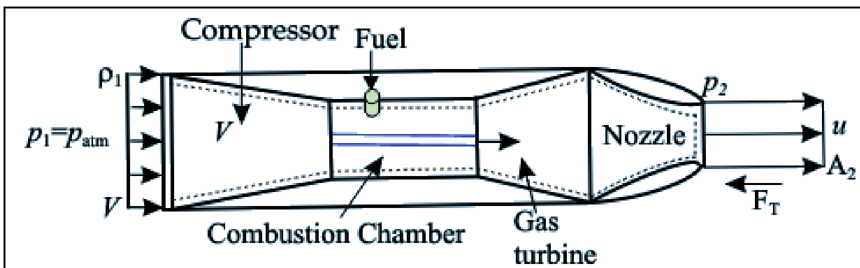


Figure 2: Turbo jet engine

Applying the momentum theorem for the C.V above as

$$(\dot{m}_a + \dot{m}_f)V_r - \dot{m}_a V = F_x - (p_2 - p_{atm})A_2$$

Or $F_x = (p_2 - p_{atm})A_2 + \dot{m}_a[(1+r)V_r - V]$ (9)

$r = \frac{mf}{ma}$

Where F_x is the force acting on the C.V along the direction of the coordinate axis .

V =is the velocity of the aircraft

$V_r = V_j - V$ is the relative velocity of the exit jet with respect to the aircraft

Vj= exit jet velocity of gas at nozzle as absolute

$\dot{m}_a \& \dot{m}_f$ Are the mass flow rate of air and mass burning rate of fuel, usually $\dot{m}_f$ is very less compared to $\dot{m}_a$. $\dot{m}_f/\dot{m}_a$ usually varies from 0.01 to 0.02 in practice. The propulsive thrust on the aircraft can be written as

$F_T = -F_x = -[(p_2 - p_{atm})A_2 + \dot{m}_a(V_r - V)]$

Since $\dot{m}_f << \dot{m}_a$, The propulsive power is given by

$P = [\,\dot{m}_a(V_r - V) + (p_2 - p_{atm})A_2]V$ (10)

The mechanical efficiency as the useful work divided by the same of useful work and kinetic energy as follows

$\eta_m = \frac{output\ power}{input\ power}$,

$\eta_m = \frac{[(p_2 - p_{atm})A_2 + \dot{m}_a(V_r - V)]V}{[(p_2 - p_{atm})A_2 + \dot{m}_a(V_r - V)]V + \frac{\dot{m}_a(Vr-V)^2}{2}}$

$if\ p_2 \approx p_{atm.}$

$\eta_m = \frac{1}{1+\frac{(V_r - V)}{2V}}$ (11)

Ex.2

An airplane consumes 1 kg fuel for each 20kg air and discharge hot gases from the tail pipe at u=1800 m/s determine the mechanical efficiency for the airplane speeds of 300 m/s & 150 m/s when $p2 \approx p_{atm}$ & $at\ V = 300\frac{m}{s}$

Sol.

At V=300m/s ; $V_r = V_j - V = 1800 - 300 = 1500$ m/s ; from Eq. 11

$\eta_m = \frac{1}{1+\frac{(V_r - V)}{2V}} = \frac{1}{1+\frac{1200}{600}} = 0.333 = 33.3\%$

$at\ V = 150\frac{m}{s};\ V_r = 1650\ m/s$

$\eta_m = 0.1666 = 16.66\%$

Ex.3

A jet engine under static test conditions in laboratory. Consumes 200 N/s air and 2 N/s fuel. What is the thrust produced from engine if the gas exit velocity is 450 m/s and the pressure at exit equal to atmosphere pressure.

Sol.

$F_t = -\dot{m}_a[(1+r)V_r - V] \quad r = \frac{2}{200} = 0.01;\ V = 0; V_r = V_j = 450\ m/s$

$F_t = -\dot{m}_a(1+r)V_r = -\frac{200}{9.81} * 1.01 * 450$

$F_t = -\ 9266\ N$

3- *Angular Momentum (Moment of Momentum).*

The moment of a force $\vec{F}$ *about O* is the vector or cross product

$$\vec{M} = \vec{r} \times \vec{F} \qquad (12)$$

Where $\vec{r}$ is the position vector from point 0 to any point on the line of action of $\vec{F}$. Vector product of two vector is a vector whose line of action is normal to the plane that contain the crossed vector ($\vec{r}$ & $\vec{F}$) from Fig.3 the magnitude of the moment of a force as

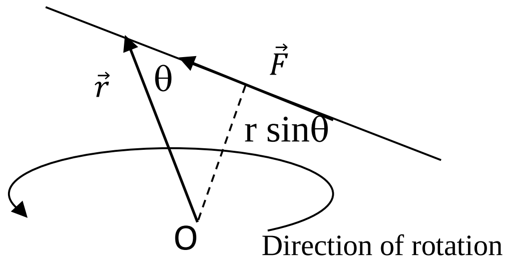


Figure 3: Moment of line force.

$$M = F\, r\, sin\theta \qquad (13)$$

Where θ is the angle between the lines of action of the vector $\vec{r}$ *and* $\vec{F}$. Replacing the vector $\vec{F}$ in Eq. 12 by the moment vector $m\vec{V}$ gives the moment of momentum, and is called the angular momentum about O as

$$\vec{H} = \vec{r} \times m\vec{V} \qquad (14)$$

The angular momentum of differential mass $dm = \rho\, d\forall$ is

$$d\vec{H} = (\vec{r} \times \vec{V})\rho\, d\forall$$

$$momentum\ of\ sys.\ \vec{H}_{sys} = \int_{sys}(\vec{r} \times \vec{V})\rho\, d\forall$$

$\therefore \vec{r} \times \vec{V}$ *is the angular momentum per unit mass*

The general C.V formulatims of the angular momentums is obtained from Eq. 1 in L-12 by setting $N = \vec{H};\ \eta = \vec{r} \times \vec{V}$ in the general Reynolds Transport Theorem. Rate of change of moment of momentum as

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{sys}(\vec{r} \times \vec{V})\rho d\forall \qquad (15)$$

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV}(\vec{r} \times \vec{V})\rho\, d\forall + \int_{CS}(\vec{r} \times \vec{V})\rho(\vec{V}_r \,.\, \vec{n})dA$$

In general $\sum \vec{M} = \frac{d}{dt}\int_{CV}(\vec{r} \times \vec{V})\rho\, d\forall + \int_{CS}(\vec{r} \times \vec{V})\rho(\vec{V}_r \,.\, \vec{n})dA \qquad (16)$

$V_r = \vec{V} - \vec{V}_{cs}$; $\rho(\vec{V}_r.\vec{n})dA$ is the mass flow rate through dA into or out the C.V. For fixed C.V $V_r = \vec{V}$

$$\sum \vec{M} = \frac{d}{dt}\int_{CV}(\vec{r} \times \vec{V})\rho\, dV + \int_{CS}(\vec{r} \times \vec{V})\rho(\vec{V} \,.\, \vec{n})dA$$

For steady flow

$$\sum \vec{M} = \int_{CS}(\vec{r} \times \vec{V})\rho(\vec{V} \,.\, \vec{n})dA \qquad (17)$$

The angular momentum flow rate can be expressed as the difference between the angular momentum of outgoing and incoming streams. If the flow is steady as well as uniform the angular momentum is

$$\sum \vec{M} = \sum_{out}(\vec{r} \times \vec{V})\dot{m} - \sum_{in}(\vec{r} \times \vec{V})\dot{m} \qquad (18)$$

In many problem, all the significant force and momentum flows are in the same plane, and then giving rise to moments in the same plane, Eq.18 can be expressed in scalar from as

$$\sum M = \sum_{out} r\, \dot{m}V - \sum_{in} r\dot{m}V \qquad (19)$$

Where r represents the normal distance between the point about which moments are taken and the line of action of the force as velocity,

Ex.4

A small lawn sprinkler operates as indicated in figure. The inlet mass flow rate is 9.98 kg/min with inlet pressure of 30 kPa. The two exit jets direct flow at an angle of 40° above the horizontal. Determine the following

a) Jet velocity relative to the nozzle.
b) Torque required to hold the arm stationary.
c) Friction torque if the arm is rotating at 35 r.p.m.
d) Maximum rotational speed if we neglect friction

Sol.

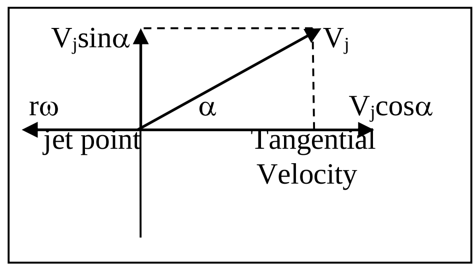
a) r_2=160 mm, d_j=5 mm

For each of the two jets :

$$Q_j = 0.5\frac{m_T}{\rho} = \frac{0.5*9.98}{1000} = 5*10^{-3}\ m^3/\min$$

$$A_j = \frac{\pi d_j^2}{4} = \frac{\pi(0.005)^2}{4} = 1.963*10^{-5}\ m^2$$

$$V_j = \frac{Q_j}{A_j} = \frac{5*10^{-3}}{60(1.963*10^{-5})} = 4.244 m/s$$



b) Torque required to hold the arm stationary taking the moment about the center of rotation

Inlet moment =0 due to r=0 the basic equation

$$\sum M = T_o = \sum_{out} \dot{m}_e(\vec{r}\times\vec{V}r) - \sum_{in}\dot{m}_i(\vec{r}\times\vec{V}r)$$

$$\therefore T_o = 2\dot{m}_e\, r(V_j\cos\propto\ - r\omega)$$

For stationary arm $r\omega$=0

$$T_o = 2\,\rho Q_j\ r\ V_j\cos\propto$$

$$\therefore T_o = 2*1000*\left(\frac{0.005}{60}\right)*0.16*4.244*cos40$$

$T_o = 0.0866\,N.m$ counter clockwise (A resisting torque which must be applied in the counterclockwise direction to keep the arm from rotating in the clockwise direction).

c) At ω=30 rpm, calculate the friction torque T_f

$$\omega = \frac{2\pi}{60}*30 = \pi\frac{rad}{s}$$

$$T_o = 2\,\rho Q_j\ r\ (V_j\cos\propto - r\omega)$$

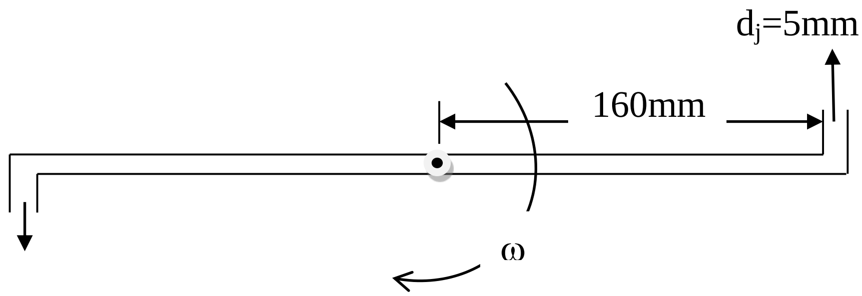
$$T_o = 2*1000*\left(\frac{0.005}{60}\right)*0.16(4.244cos40 - \pi*0.16)$$

$$T_o = 0.07329\,N.m$$

d) The maximum rotational speed occurs when the opposing torque is zero and all the moment of momentum goes to the angular rotation.

$V_j cos \propto -r\omega = 0$

$\omega = \frac{V_j cos\propto}{r} = \frac{4.244*cos40}{0.16} = 20.319\,\frac{rad}{sec} = 194\ rpm$



4- *Radial-Flow Devices.*

The fluid will be affected by centrifugal action of moving blades from the inner radius to the outer radius. Due to the suction created by the impeller motion, the fluid enters the eye of the impeller axially. The momentum transfer to the fluid by the impeller blades will increase the total head of the fluid and causing the fluid to flow out.

To analyze the centrifugal pump, we choose the annular region that encloses the impeller section as the *C.V*

ω: is the angular velocity of shaft impeller blades will have a tangential velocity

$V_{1,t}=\omega r_1$ at the inlet
$V_{2,t}=\omega r_1$ at the outlet (20)

For steady incompressible flow, the conservation of mass equation can be written as

$\dot{Q}_1 = \dot{Q}_2 = \dot{Q} ---\rightarrow (2\pi r_1 b_1)V_{1,n} = (2\pi\, r_2 b_2)V_{2,n}$ (21)

Where b_1&b_2 are the flow widths at $r=r_1$ inlet & $r=r_2$ at outlet

The average normal components $V_{1,n}$ & $V_{2,n}$ of absolute velocity can be expressed in terms of the volumetric flow rate Q as

$V_{1,n} = \frac{Q}{2\pi r_1 b_1}$ & $V_{2,n} = \frac{Q}{2\pi r_2\, b_2}$ (22)

Since $V_{1,n}$ **&** $V_{2,n}$ pass through the shaft center , thus they do not contribute to torque about the origin. Only the tangential velocity components contribute to torque and the application of the angular momentum as

$\sum M = \sum_{out} r\,\dot{m}V - \sum_{in} r\dot{m}V$ (23)

$\sum T_{shaft} = \dot{m}\left(r_2 V_{2,t} - r_1 V_{1,t}\right)$ (24)

Is known as Euler's turbine formula from Fig. 4.

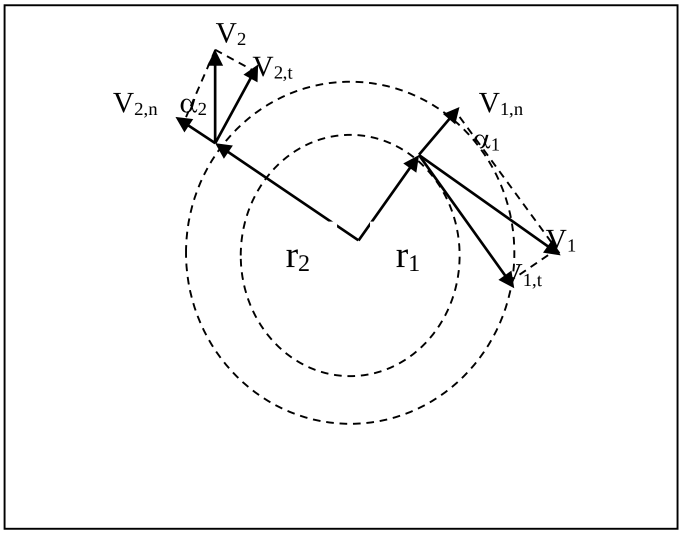


Figure 4: Velocities components in radial flow

$\propto_2$ & $\propto_1$ Angles between the direction of absolute flow velocities & the radial direction. Substituting Eq.20 in Eq.24 gives an ideal case when the tangential velocity begin equal to the blade angular velocity both at inlet & outlet

$T_{sh,ideal} = \dot{m}\,\omega(r_2^2 - r_1^2)$

Shaft power $P_{sh} = \omega\, T_{sh} = \frac{2\pi n}{60}\, T_{sh}$ (25)

Ex.5

Centrifugal blower has the following specifications.

$r_1 = 20cm\,, b_1 = 8.2cm \;\; at\ inlet$

$r_2 = 45cm\;\,, b_2 = 5.6cm\ at\ outlet$

$Q = 0.7\frac{m^3}{s}\;,\; n = 700\ r.p.m$

$\propto_1 = 0°\ at\ inlet \quad\quad \propto_2 = 50°\ from\ radial\ direction$

Determine the minimum power consumption of the blower $\rho_{air} = 1.25kg/m^3$

Sol.

$\sum \;\; M = \sum_{out} r\,\dot{m}\vec{V} - \sum_{in} r\,\dot{m}\vec{V}$

$Q_1 = Q_2 = Q = 0.7\ m^3/s\,,\quad \dot{m} = \rho * Q = 1.25 * 0.7 = 0.875\ kg/s$

$V_2 = \frac{Q}{A2} = \frac{0.7}{(2\pi r2 * b2)} = \frac{0.7}{(2\pi * 0.45 * 0.056)} = 4.42\frac{m}{s}$

$T_{sh} = \dot{m}(r_2\, V_2 sin \propto 2 - r_1 V_1 sin \propto 1)$

$= 0.875(0.45 * 4.42 * sin50 - 0) = 1.33 \quad N.m$

$P = \omega.T_{sh} = \frac{2\pi n}{60}\; T_{sh} = \frac{2\pi * 700}{60} * 1.33 = 97.75\ W$