Lecture Twelve
Momentum Equation

1- *Conservation of Momentum*.

In Newtonian mechanics, the conservation of momentum is defined by Newton's second law of motion. The 2nd law of motion states as

- The rate of change of momentum of a body is proportional to the impressed action and takes place in the direction of the impressed action.
- If a force acts on the body, linear momentum is implied.
- If a torque (moment) acts on the body angular momentum is implied.

Statement of Reynolds Transport Theorem, "*the time rate of increase of property (N) within a control mass system (CMS) is equal to the time rate of increase of property (N) within the control volume (CV) plus the net rate of efflux of the property (N) across the control surface* (CS)".

$$\left(\frac{dN}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{C.V}\eta\,\rho d\forall + \iint_{C.V}\eta\rho\,\vec{V_r}\,d\vec{A} \qquad (1)$$

Eq. 4.19 is known as Reynolds Transport Theorem, $\vec{V_r} = \vec{V} - \vec{V_c}$

$\vec{V_r}$= fluid velocity relative to *C.V*.

$\vec{V}$&$\vec{V_c}$ = velocities of fluid & *C.V*. as observed in a fixed frame reference.

N= flow property which is transported.

η= intensive value of the flow property.

2- *Linear Momentum.*

From Eq. 1, the property *N* as in the linear- momentum m$\vec{V}$ & η as the velocity $\vec{V}$. Then it becomes

$$\frac{d}{dt}\left(m\vec{V_r}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV}(\vec{V_r})\rho\ d\forall + \iint_{CS}\vec{V_r}\ \rho\left(\vec{V_r}.d\vec{A}\right) \qquad (2)$$

Where $\vec{V_r}$ is the velocity defining the linear momentum in above equation. L.H.S of Eq.2 represents the external forces $\sum\vec{F}$ on the *CMS* or on the coinciding *C.V* by the direct application of Newton's law of motion. The L.H.S of Eq. 2 form Newton's law of motion can be written as

$$m\left(\frac{d\vec{V}}{dt}\right)_{CMS} = \sum\vec{F}$$

Therefore

$$\left(m\frac{d\vec{V_r}}{dt}\right)_{CMS} = m\left(\frac{d\vec{V_r}}{dt}\right)_{CMS} = m\frac{d}{dt}\left(\vec{V}-\vec{V_C}\right)_{CMS} = m\left(\frac{d\vec{V}}{dt}\right)_{CMS} - m\vec{a}_C$$

Where $a_c = \left(\frac{d\vec{V}_C}{dt}\right)$ is the rectilinear acceleration of the *C.V* (observed in a fixed coordinate system).

Therefore, $m\left(\frac{d\vec{V_r}}{dt}\right)_{CMS} = \sum\vec{F} - m\vec{a}_C \qquad (3)$

Eq. 2 can be written as follows after consider Eq. 3

$$\sum\vec{F} - m\vec{a}_C = \frac{\partial}{\partial t}\iiint_{CV}(\vec{V_r})\rho\ d\forall + \iint_{CS}\vec{V_r}\ \rho\left(\vec{V_r}.d\vec{A}\right) \qquad (4)$$

At steady state form it becomes

$$\sum\vec{F} - m\vec{a}_C = \iint_{CS}\vec{V_r}\ \rho\left(\vec{V_r}.d\vec{A}\right) \qquad (5)$$

In case of an inertial C.V, which is either fixed or moving with a constant rectilinear velocity $\vec{a}_c = 0$, Now, Eq's. (4 & 5) becomes

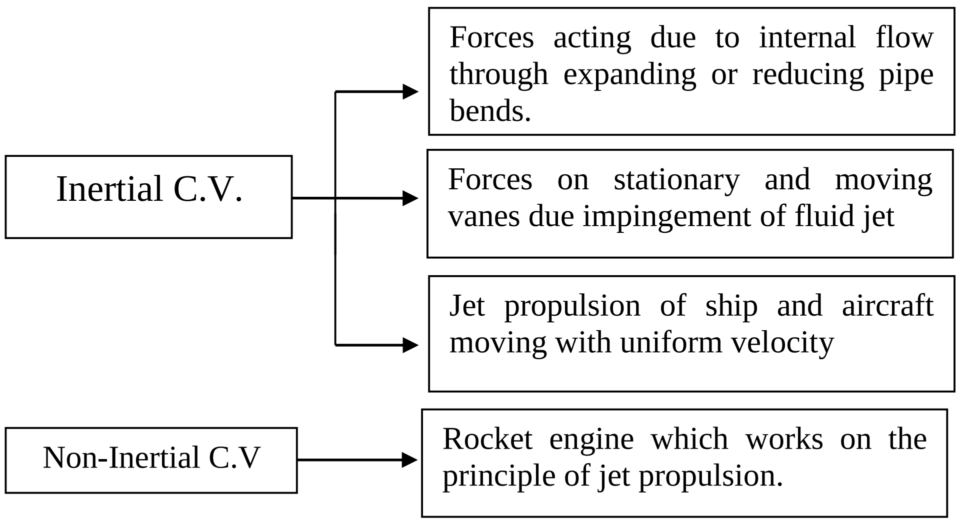
$$\sum\vec{F} = \iint_{CS}\vec{V_r}\ \rho\left(\vec{V_r}.d\vec{A}\right) \qquad (6)$$

Or $\sum \vec{F} = \frac{\partial}{\partial t} \iiint_{CV} (\vec{V}_r) \rho \; d\forall + \iint_{CS} \vec{V}_r \; \rho \left(\vec{V}_r . d\vec{A} \right)$ (7)

Eq's (4 & 5) for non-inertial C.V having an arbitrary rectilinear acceleration.

3- *The Application of Momentum Theorem.*

From the conservation of momentum phenomenon we can state the law of conservation of momentum as follows " the net force acting on a mass of fluid is equal to change in momentum of flow per unit time in that direction". The application of momentum theorem in some practical cases of inertial and non-inertial *C.V* can be treated. Three distinct types of practical problems for inertial *C.V* namely



Liner momentum of *C.V* in a system is (m.$\vec{V}$) from Newton's 2nd law

$$\sum \vec{F} = \frac{d(m\vec{V})}{dt} = \frac{\partial}{\partial t} \int_{C.V} \rho \, \vec{V} \; d\forall + \int_{C.S} \rho \vec{V} \, (\vec{V} . d\vec{A})$$

(i.e) the resultant force acting on a *C.V* is equal to the time rate of increase of linear momentum within the *C.V* plus the net output of linear momentum from the *C.V*. Types of forces acting on control volume (*C.V*)

- Body force the weight of fluid
- Pressure force
- Hydrostatic force
- Shear force

$$\sum \vec{F}_{total} = \sum \vec{F}_{gravity} + \sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} \quad (8)$$

$$\sum \vec{F} = m\vec{a} = m \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

Where $(m\vec{V})$ is the linear momentum of system. Now the 2nd law can be expressed more general as

$\sum \vec{F} = \frac{d}{dt} \int_{sys} \rho \, \vec{V} \, d\forall$ where $dm = \rho d\forall$

External force equal to the time rate of change of linear momentum of system. If system at rest or move with constant velocity then from Reynolds Transport Theorem is applied on *C.V* formulation as follows

$$\frac{d(m\vec{V})_{sys}}{dt} = \frac{\partial}{\partial t} \iiint_{CV} \rho (\vec{V}) \; d\forall + \iint_{CS} \rho \;\; \vec{V} \;\; \left(\vec{V}_r . \vec{n}\right) dA \quad (9)$$

But the left side of Eq. 9 is equal to $\sum \vec{F}$

$$\sum \vec{F} = \frac{\partial}{\partial t} \iiint_{CV} \rho(\vec{V}\ )\ d\forall + \iint_{CS} \rho\ \ \vec{V}\ \ (\vec{V}_r.\vec{n})dA \qquad (10)$$

Eq. 10 can be expressed as

(The sum of all external force acting on C.V)= (The times rate of change of linear momentum of the contents of the C.V) + (The net flow rate of LM out of the C.V by mass flow).

Here $\vec{V}_r = \vec{V} - \vec{V}_{cs}$ is the fluid velocity relative to the *C.V*. $\vec{V}$ is the velocity of fluid as viewed from fixed reference frame. The product $\rho(\vec{V}_r.\vec{n})dA$ represents the mass flow rate through area element dA into or out of the *C.V* for fixed *C.V*. For fixed *C.V* no motion of *C.V* or deformation $V_r = V$ and the linear- momentum equation for fixed *C.V* becomes

$$\sum \vec{F} = \frac{d}{dt} \int_{C.V} \rho\, \vec{V}\, d\forall + \int_{CS} \rho\, \vec{V}\, (\vec{V}.\vec{n})dA \qquad (11)$$

For steady the derivative with respect to time is equal to zero

$$\sum \vec{F} = \int \int_{CS} \rho\, \vec{V}\, (\vec{V}.\vec{n})dA \qquad (12)$$

Mass flow rate across an inlet or outlet

$$\dot{m} = \int_{Ac} \rho(\vec{V}.\vec{n})dA_c = \rho V_m\, A_c \qquad (13)$$

∴Momentum flow rate across inlet or outlet

$$\int_{Ac} \rho\, \vec{V}(\vec{V}\,.\vec{n})dA_c = \rho V_m A_c\, \vec{V}_m = \dot{m}\, \vec{V}_m \qquad (14)$$

V_m= uniform mean velocity

$$\sum \vec{F} = \sum_{out} \dot{m}\vec{V} - \sum_{in} \dot{m}\vec{V} \qquad (15)$$

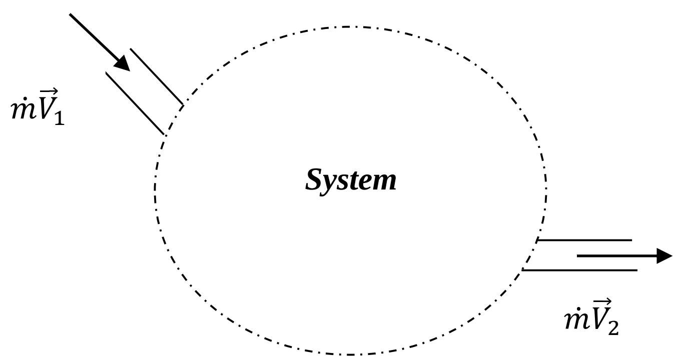


Figure 1: Linear momentum of system.

Along x-direction as in Fig.1

$$\sum \vec{F}_x = \dot{m}(\vec{V}_{2x} - \vec{V}_{1x})$$

Similarly $\sum F_y = \rho Q(V_{2y} - V_{1y}) = \dot{m}(V_{2y} - V_{1y})$

For any C.V the total force $\vec{F}$ which acts upon it in a given direction will made up of three components $\vec{F} = F1 + F2 + F3$

- F1=force exerted in the direction on the fluid in the *C.V* by any solid body within the C.V or coinciding with boundaries of the *C.V*.
- F2=force exerted in the given direction on the fluid in the C.V by body for such as gravity.
- F3=force exerted in the give direction of fluid in the C.V by the fluid outside the C.V such as pressure.

The effects of these forces on *C.V* can be study through practical engineering problem from the following cases.

I- Forces due to Flow through Expanding or Reducing Pipe Bends.

Fig's.2 and 3 shows the fluid flow through an expander where F_x &F_y are the external forces on the fluid areas 2-3 & 1-4 due to net efflux linear momentum through the interior surface of the expander since *C.V* (1 2 3 4) is stationary and at steady state apply Eq.6 for *x & y* components.

$\dot{m}V_2\,cos\theta - \dot{m}V_1 = p_1A_1 - p_2A_2\,cos\theta + F_x$

And $\quad \dot{m}V_2\,sin\theta - 0 = -p_2A_2sin\theta + F_y - mg$

Or $\quad F_x = \dot{m}(V_2\,cos\theta - V1) + p_2A_2\,cos\theta - p_1A_1 \qquad (16)$

$F_y = \dot{m}V_2sin\theta + p_2A_2\,sin\theta + mg \qquad (17)$

Where

$\dot{m}$=mass flow rate through the expander, analytically it can be expressed as

$\dot{m} = \rho A_1V_1 = \rho A_2V_2$

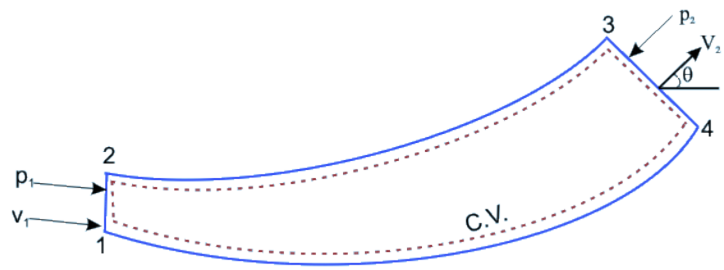


Figure 2: Flow of a fluid through an expander.

m represents the mass of fluid contained in the expander at any instant and can be expressed as $m = \rho\forall$ Where $\forall$ internal volume of the expander, F_x & F_y forces acting on the *C.V* by the expander. According to Newton's third law (any action there is a reaction) the expander will experience the forces

$R_x = -F_x$ & $R_y = -F_y$ are the reactions in the x&y directions respectively as shown in the free body diagram of the expander Fig.4.

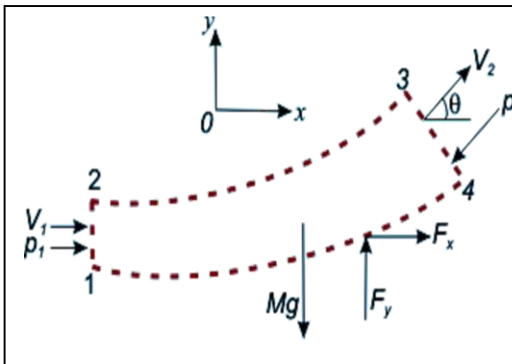


Figure 3: Control volume comprising the fluid contained in the expander at any instant.

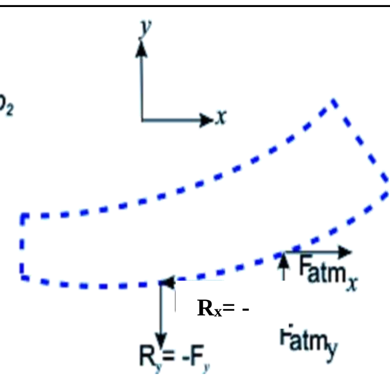


Figure 4: Free body diagram of the expander.

The magnitude of the resultant force acting on the pipe bend is

$$R = \sqrt{R_x^2 + R_y^2}$$

And the direction of the resultant force with x-axis is

$$\theta = tan^{-1}\left(\frac{R_y}{R_x}\right)$$

Ex.6

A 600 mm diameter pipeline carries water under a head of 30 m with velocity of 3 m/s. This water main is fitted with a horizontal bend which turns the axis of the pipeline through 75°. Calculate the resultant force on the bend and its angle to the horizontal.

Sol.

$$A_1 = A_2 = \pi\left(\frac{0.6}{2}\right)^2 = 0.283\, m^2$$

$$d = 0.6m, h = 30m$$

$$V_1 = V_2 = 3\frac{m}{s}, p_1 = \rho g h = 9810 * 30 = 294300\, {}^{N}/_{m^2}$$

$$Q = A_1V_1 = A_2V_2 = 0.283 * 3 = 0.849\, {}^{m^3}/_{s}$$

From Eq.(4.34 & 4.35) $\dot{m} = \rho Q = 1000 * 0.849 = 849\frac{kg}{s}$

$$F_x = \dot{m}(V_2\, cos\theta - V1) + p_2A_2\, cos\theta - p_1A_1$$

$$F_x = 849(3 * cos75 - 3) + 294300 * 0.283 * cos75 - 294300 * 0.283$$

$$F_x = -63.618\, kN$$

$$F_y = \dot{m}V_2 sin\theta + p_2A_2\, sin\theta + mg$$

$= 849 * 3 * sin75 + 294300 * 0.283 * sin75$ Since the bend in horizontal

$$Fy = 82.9\, kN$$

$$\therefore\ R_x = -F_x = +63.618\, kN,\ R_y = -F_y = -82.9kN$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(63.618)^2 + (-82.9)^2} = 104.5\, kN$$

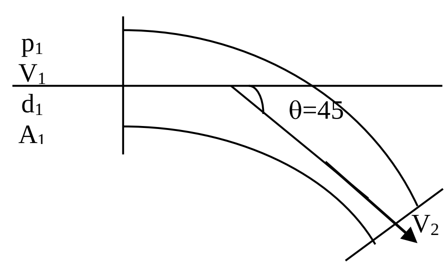
$$\theta = tan^{-1}\left(\frac{R_y}{R_x}\right) = tan^{-1}\left(\frac{-82.9}{63.618}\right) = -52.5^{\circ}$$

Ex.7

A pipe bend tapers from a diameter d, of (500) mm at inlet to a diameter d2 of (250mm) at outlet and turns the flow through an angle (θ) of 45°. Measurements of (p_1 & p_2) at inlet and outlet are 40 kN/m^2 and 23 kN/m^2. If the pipe is conveying oil which has a density ρ=850 kg/m^3. Calculate the magnitude and direction of resultant force on the bend when the oil is flowing at the rate of 0.45 m^3/s. The bend is in a horizontal plan. (Gravity force=0)

Sol.

$$\dot{m} = \rho AV = \rho Q = 850 * 0.45 = 382.5\, kg/s$$



$$F_x = \dot{m}(V_{2x} - V_{1x}) + p_1A_1 - p_2A_2\, cos\theta$$

$$V_2 = \frac{\boldsymbol{Q}}{\boldsymbol{A_2}} = 0.45 * \frac{4}{\pi(0.25)^2} = 9.16\,\mathrm{m/s}$$

$$V_1 = \frac{Q}{A_1} = 0.45 * \frac{4}{\pi(0.5)^2} = 2.29\ \mathrm{m/s}$$

$$F_x = 382.5(9.16\cos 45 - 2.29) + 40 * 10^3 * \frac{\pi(0.5)^2}{4} - 23 * 10^3 * \frac{\pi(0.25)^2}{4}\cos 45$$

$\therefore F_x = 8657 \frac{N}{m^2} \quad --\rightarrow \mathrm{R_x} = -8657\ N/m^2$

$F_y = \dot{m}\left(V_{2y} - V_{1y}\right) + p_2 A_2\ sin\theta = 382.5(9.16 \sin 45 - 0) + 23 * 10^3 * sin\ 45 = 18740.9\ N/m^2$

$R_y = -F_y = -18740.9\ N/m^2$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-8657)^2 + (-18740.9)^2} = 20643\ N/m^2$$

$\theta = tan^{-1} \frac{R_y}{R_x} = 65^\circ$