Lecture Eleven
Energy Equation

1- Energy Equation of an Ideal Flow along a Stream Line.

Derivation of Bernoulli's Equation.

Euler's equation along a streamline is derived by applying Newton's second law of motion to fluid element moving along a stream line. Considering gravity as the only the body force component acting vertically downward, the net external force acting on the fluid element moving along the direction of stream line as shown in Fig.1, the equation of motion given as

$$\sum F_s = ma_s \quad (1)$$

Take the velocity function of s & t , V(s , t)

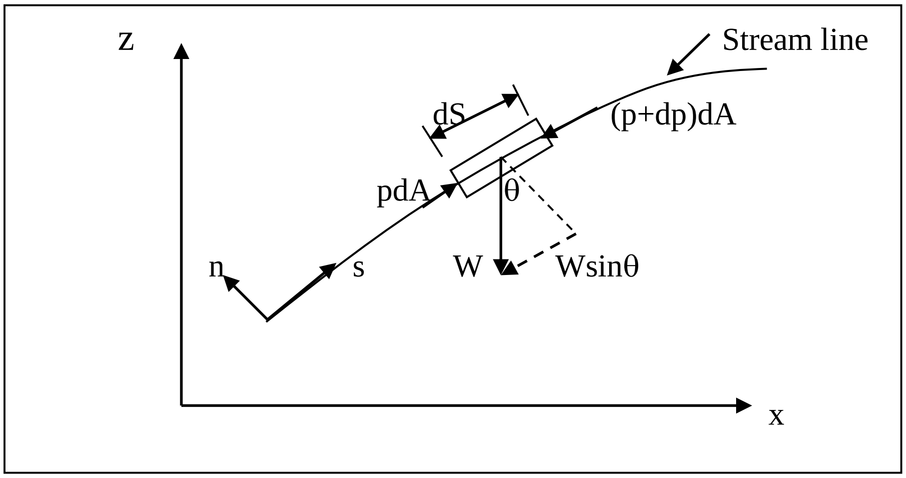


Figure 8: Fluid element moving along stream line.

Total differential of V(s,t)is

$$dV = \frac{\partial V}{\partial s}\, ds + \frac{\partial V}{\partial t}\, dt \qquad \text{Divided by dt}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t}$$

In steady flow $\frac{\partial V}{\partial t} = 0$

Then V=V(s) , the acceleration in the s-direction becomes

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s}\, V = V\frac{dV}{ds} \quad (2)$$

The forces acting in the s-direction are the pressure and the component of particle weight in the s-direction from Eq's. (1 &2)

$$p\, dA - (p + dp)dA - W sin\theta = m\, V\frac{dV}{ds} \quad (3)$$

Where θ is the angle between the normal to the streamline and the vertical z-axis at that point.

$$m = \rho \forall = \rho\, dA\, ds$$

$$W = mg = \rho g\, dAds$$

$$sin\theta = \frac{dz}{ds}$$

Substituting in Eq.3

$$-dp\ dA - \rho g\ dA\ ds \frac{dz}{ds} = \rho\ dA\ ds\ V \frac{dV}{ds} \qquad (4)$$

Divided by dA and simplifying

$$-dp - \rho g\ dz = \rho V\ dV$$

Note V dV $= \frac{1}{2}$ d(V^2)& dividing each term by ρ gives

$$\frac{dp}{\rho} + \frac{1}{2}\ d(V^2) + \ g\ dz = 0 \qquad (5)$$

Integrating along streamline

$$\int \left(\frac{dp}{\rho} + \frac{1}{2}\ d(V^2) + \ g\ dz\right) = constant$$

For, incompressible ρ= const.

$$\frac{p}{\rho} + \frac{V^2}{2} + \ gz = constant$$

This is the famous Bernoulli's equation *B.E* used for steady incompressible & inviscid regions flow. The *B.E* can be written between any two points on the same streamline as

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + \ gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + \ gz_2 \qquad (6)$$

Eq. 6 can be written as

$$\frac{p}{\rho} + \frac{V^2}{2} + \ gz = C \qquad (7)$$

Where *C* is a constant along a streamline. In case of an incompressible flow, Eq. 4.16 is based on the assumption no work or heat interaction between a fluid element and the surrounding take place, its terms illustrate as follows,

- 1st term represents the flow work per unit mass.
- 2nd term represents the kinetic energy per unit mass.
- 3rd term represents the potential energy per unit mass.

The sum of three terms represents the total mechanical energy per unit mass which remains constant along a streamline for steady, invicid & incompressible flow of fluid. Eq.7 is known as mechanical energy equation. Also, Eq. 7 can be expressed in terms of energy per unit weight as

$$\frac{p}{\rho g} + \frac{V^2}{2g} + \ z = C1 \qquad (8)$$

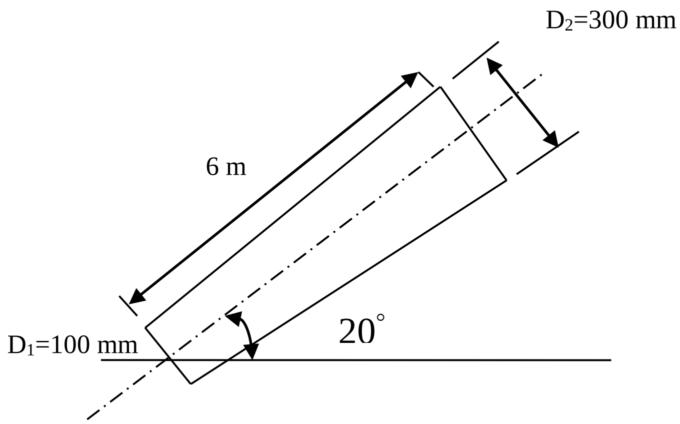
The energy per unit weight is termed as a ***Head***, Eq.8 can be written as
(Pressure head)+(Velocity head)+ (Potential head)= Total head
In many practical situations, problems related to real fluid and can be analyzed with help of a modified form of Bernoulli's equation as

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + \ z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + \ z_2 + h_f \qquad (9)$$

Where h_f represents the frictional work done (the work done against the fluid friction) per unit weight of fluid element.

<u>Ex.4</u>

A 6m long pipe is inclined at angle of 20° with the horizontal. The smaller section of the pipe which is at lower level is of 100mm and the larger section of pipe is of 300 mm diameter as shown in figure. If the pipe is uniformly tapering and the velocity of water at the smaller section is 1.8 m/s determine the difference of pressures between the two sections.



Sol.

$A_1 = \frac{\pi d1^2}{4} = \pi * \frac{0.1^2}{4} = 0.00785 m^2$

$V_1 = 1.8 \frac{m}{s}$

$z_1 = 0\ m$

$d_2 = 0.3m$

$A_2 = \frac{\pi}{4} * 0.3^2 = 0.0707 m^2$

$z_2 = 6 sin20 = 6 * 0.342 = 2.05m$

From *C.E* $A_1 V_1 = A_2 V_2$

$\therefore V_2 = \frac{A_1 V_1}{A_2} = 0.00785 * \frac{1.8}{0.0707} = 0.2\ m/s$

Applying B.E. to both sections of pipe

$\frac{p1}{\gamma} + \frac{V1^2}{2g} + z_1 = \frac{p2}{\gamma} + \frac{V2^2}{2g} + z_2$

$p_1 - p_2 = \gamma \left(\frac{V_2^2 - V_1^2}{2g} + z_2 \right)$

$p_1 - p_2 = 9810 \left(\frac{0.2^2 - 1.8^2}{2*9.81} + 2.05 \right) = 18510 \frac{N}{m^2}$

Ex.5

a) Determine the velocity of efflux from the nozzle in the wall of the reservoir as in figure.
b) Find the discharge at the nozzle.

Sol.

a) $\frac{V_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\rho g} + z_2$

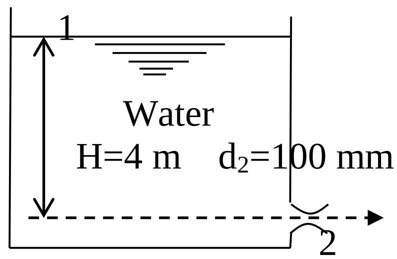
With pressure datum as local atmospheric pressure, p_1=p_2=0 & z_2=0, z_1=H, the velocity on the surface of the reservoir is zero Hence .

$$0 + 0 + H = \frac{V_2^2}{2g} + 0 + 0$$

$V_2 = \sqrt{2gH} = \sqrt{2 * 9.81 * 4} = 8.86 \frac{m}{s}$

This is known as torricellis theorem

b) $Q = A_2 V_2 = \pi (0.05)^2 (8.86) = 0.07 \frac{m^3}{S} = 70\ \frac{L}{S}$


1
Water
H=4 m
d2=100 mm
2