Lecture Ten
Fluid Dynamics

1- ***Introduction.***

In previous chapters the forces exerted by static fluid on stationary surfaces was discussed. In this chapter the forces exerted by moving fluid particles in the flow field is discussed. In many cases the surfaces or the cross section area cause a variations or change in the magnitude and direction of the fluid particles velocity in the flow field, the fluid particles exert a force on the surface. In opposite direction the surface exert an equal force on the fluid particles. *The force exerted by moving fluid particles on the surface is called dynamic force*.

2- ***Definitions.***

System:- A quantity of matter in space which is analyzed during problem.
Surrounding:- Everything external to the system .
System Boundary:- A separation present between system and surrounding.
Classifications of the system boundary:-
Real solid boundary and imaginary boundary.
The system boundary may be further classified as:-

- Fixed boundary as control mass system.
- Moving boundary as control volume system.

The choice of boundary depends on the problem being analyzed.

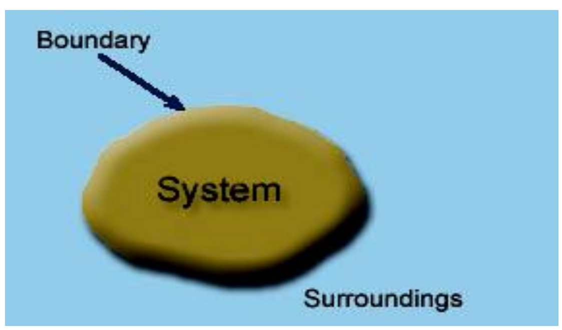
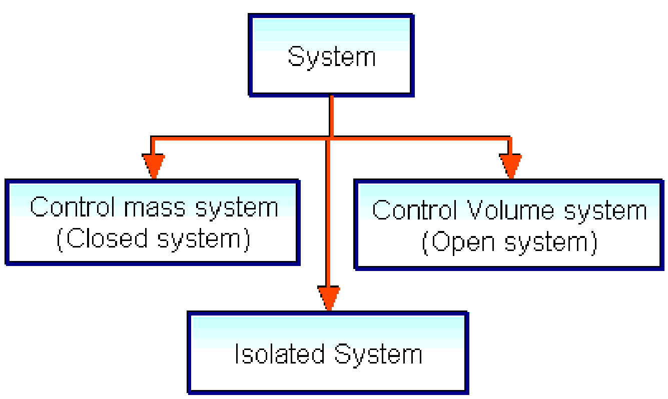


Figure 1: System and surroundings.

Classification of Systems.



3- *Types of System.*

a- Control Mass System (Closed System)

1. It's a system of *fixed mass* with *fixed identity*.
2. This type of system is usually referred to as *"closed system"*.
3. There is no mass transfer across the system boundary.
4. Energy transfer may take place into or out of the system as in Fig.2.

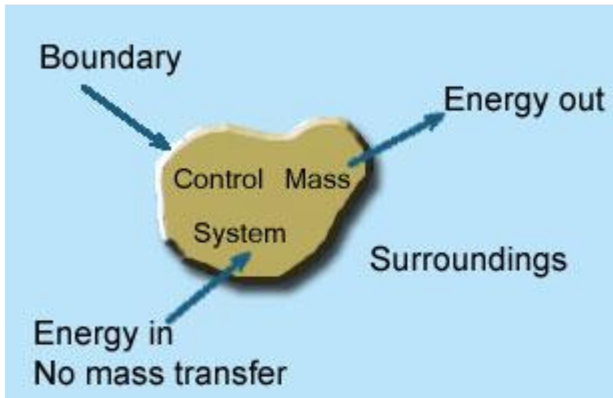


Figure 2: A control mass system or closed system.

b- Control Volume System (Open System)

1. It's a system of fixed *volume*.
2. This type of system is usually referred to as *"open system"* or a *"control volume" C.V.* as in Fig.3.
3. Mass transfer can take place across a control volume.
4. Energy transfer may also occur into or out of the system.
5. A control volume can be seen as a fixed region across which mass and energy transfers are studied.
6. Control Surface- It's the boundary of a control volume across which the transfer of both mass and energy takes place.
7. The mass of a control volume (open system) may or may not be fixed.
8. When the net influx of mass across the control surface equals zero then the mass of the system is fixed and vice-versa.
9. The identity of mass in a control volume always changes unlike the case for a control mass system (closed system).
10. Most of the engineering devices, in general, represent an open system or control volume.

Examples.

- Heat exchanger - Fluid enters and leaves the system continuously with the transfer of heat across the system boundary.
- Pump - A continuous flow of fluid takes place through the system with a transfer of mechanical energy from the surroundings to the system.

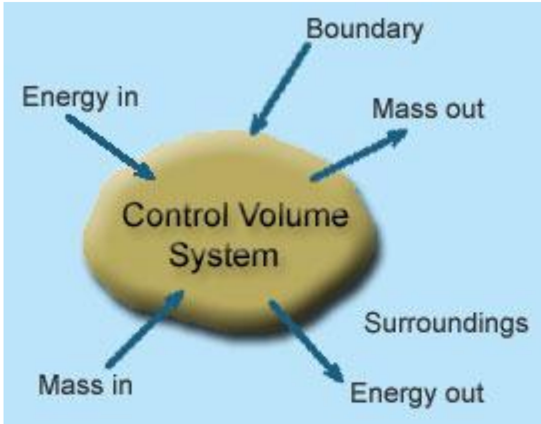


Figure 3: A control volume system or open system.

c- Isolated System.

1. It's a system of *fixed mass* with same *identity and fixed energy*.
2. No interaction of mass or energy takes place between the system and the surroundings as in Fig.4.
3. In more informal words an isolated system is like a closed shop amidst a busy market.

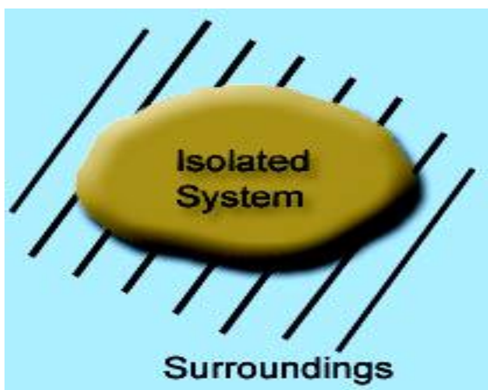


Figure 4: An isolated system.

4- ***Basic Laws.***

1- Law of mass conservation.
2- Law of momentum conservation.
3- Law of Energy conservation.

There are two method of derivation for each law.

A- Use of differential element, and then by integration for more than one dimension.
B- Use of free body, used for one-dimension without need to integration.

A- ***Conservation of Mass - The Continuity Equation.***

Law of conservation of mass states that *mass can neither be created nor be destroyed*. Conservation of mass is inherent to a control mass system (closed system).

- The mathematical expression for the above law is stated as:
  $\Delta m/\Delta t = 0$, where m = mass of the system
- For a control volume Fig.5, the principle of conservation of mass is stated as

Rate at which mass enters = Rate at which mass leaves the region + Rate of accumulation of mass in the region

Or

Rate of accumulation of mass in the control volume + Net rate of mass efflux from the control volume = 0 (1)

The above statement expressed analytically in terms of velocity and density field of a flow is known as the *continuity equation C.E.*

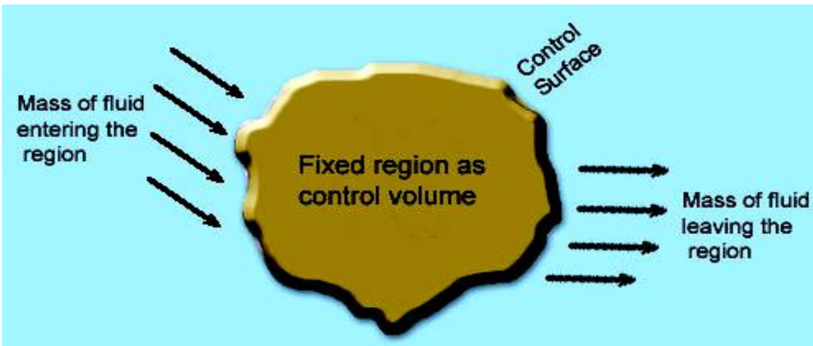


Figure 5: A control volume in a flow field.

B- Continuity Equation - Differential Form.

1. The point at which the continuity equation has to be derived, is enclosed by an elementary control volume.
2. The influx, efflux and the rate of accumulation of mass is calculated across each surface within the control volume.

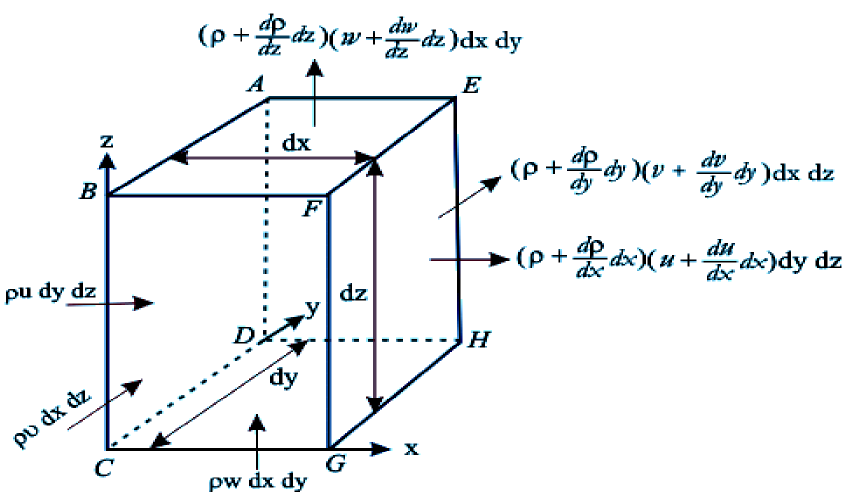


Figure 6: A Control volume appropriate to a rectangular cartesian coordinate system.

Consider a rectangular parallelepiped in the above figure as the control volume in a rectangular cartesian frame of coordinate axes.

- Net efflux of mass along x -axis must be the excess outflow over inflow across faces normal to x -axis.
- Let the fluid enter across one of such faces ABCD with a velocity u and a density ρ. The velocity and density with which the fluid will leave the face EFGH will be (neglecting the higher order terms in δx).

$$u+\frac{\partial u}{\partial x}\,dx \;\&\; \rho+\frac{\partial \rho}{\partial x}\,dx$$

The rate of mass entering the C.V through ABCD= ρu dy dz (a)

Therefore the rate of mass leaving the face EFGH will be

$$=\left(\rho+\frac{\partial \rho}{\partial x}\,dx\right)\left(u+\frac{\partial u}{\partial x}dx\right)dydz$$

$$=\left[\rho u+\rho\frac{\partial u}{\partial x}\,dx+u\frac{\partial \rho}{\partial x}\,dx+\frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}(dx)^2\right]dydz$$

Neglecting the higher order terms in (dx)

$$=\left(\rho u+\frac{\partial}{\partial x}\,(\rho u)dx\right)dydz \qquad (a')$$

Similarly influx and efflux take place in all y and z directions also.

The rate of mass entering the C.V through (BCGF) =ρv dx dz (b)

The velocity & density when the fluid leaves the face (AEHD) will be

$\left(v+\frac{\partial v}{dy}dy\right)\&\left(\rho+\frac{\partial \rho}{\partial y}\,dy\right)$ {Neglecting higher order} therefore rate of mass leaving the face (AEHD) will be

$$=\left[\left(\rho+\frac{\partial \rho}{\partial y}\,dy\right)\left(v+\frac{\partial v}{\partial y}\,dy\right)\right]dx\,dz$$

$$=\left[\rho v+\rho\frac{\partial v}{y}\,dy+v\frac{\partial \rho}{\partial y}\,dy+\frac{\partial \rho}{\partial y}*\frac{\partial v}{\partial y}(dy)^2\right]dx\,dz$$

Neglecting the higher order terms in (dy)

$$=\left[\rho v+\frac{\partial}{\partial y}\,(\rho v)dy\right]dx\,dz \qquad (b')$$

The rate of mass entering the C.V through (CDHG)= ρw dy dx (c)

The velocity & density when the fluid leaves the face (ABFE) will be

$$(\mathrm{w}+\frac{\partial \mathrm{w}}{\partial \mathrm{z}}\mathrm{dx})\;\&\;\left(\rho+\frac{\partial \rho}{\partial \mathrm{z}}\mathrm{dz}\right)$$

Therefore the rate of mass leaving the face (ABFE) will be

$$=\left[\left(\rho+\frac{\partial \rho}{\partial z}\,dz\right)\left(w+\frac{\partial w}{\partial z}\,dz\right)\right]dy\,dx$$

$$=\left[\rho w+\rho\frac{\partial w}{\partial z}\,dz+w\frac{\partial \rho}{\partial z}\,dz+\frac{\partial \rho}{\partial z}\frac{\partial w}{\partial z}\,(dz)^2\right]dy\,dx$$

Neglecting the higher order terms in (dz)

$$=\left[\rho w+\frac{\partial}{\partial z}\,(\rho w)dz\right]dy\,dx \qquad (c')$$

Rate of accumulation for a point in a flow field

$$\frac{\partial m}{\partial t}=\frac{\partial \rho}{\partial t}\,d\forall \qquad (d)$$

Rate of Entering fluid = Rate of Accumulation fluid + Rate of leaving fluid

$$Eq.(a)+Eq.(b)+Eq.(c)=Eq.(d)+\;Eq(\acute{a}\,)+Eq\left(\acute{b}\,\right)+Eq(\acute{c}\,)$$

$$\rho u\ dy\ dz + \rho v\ dx\ dz + \rho w\ dx\ dy = \frac{\partial \rho}{\partial t}\ d\forall + \rho u\ dy\ dz + \frac{\partial}{\partial x}\ (\rho u) dx\ dy\ dz + \rho v\ dx\ dz + \frac{\partial}{\partial y}\ (\rho v) dxdy\ dz + \rho w\ dx\ dy + \frac{\partial}{\partial z}\ (\rho w) dx\ dy\ dz$$

$d\forall = dxdydz,$ Rearrangement the above equation

$$\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\ (\rho u) + \frac{\partial}{\partial y}\ (\rho v) + \frac{\partial}{\partial z}\ (\rho w)\right]\ d\forall = 0 \qquad (2)$$

This is the equation of continuity for a compressible fluid in a rectangular cartesian coordinate. The continuity equation for cylindrical polar coordinate system for a compressible fluid can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\ (\rho V_r) + \frac{\rho V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\ (\rho V_\theta) + \frac{\partial}{\partial Z}\ (\rho V_z) = 0 \qquad (3)$$

C- Continuity Equation (C.E) - Vector Form.

If $\vec{V} = u\vec{\imath} + v\vec{\jmath} + w\vec{k}$ is the velocity of the point

$$\nabla = \left(\frac{\partial}{\partial x}\ \vec{\imath} + \frac{\partial}{\partial y}\ \vec{\jmath} + \frac{\partial}{\partial z}\ \vec{k}\right)$$

$$\therefore \frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x}\ \vec{\imath} + \frac{\partial}{\partial y}\ \vec{\jmath} + \frac{\partial}{\partial z}\ \vec{k}\right).\left[\rho u\vec{\imath} + \rho v\vec{\jmath} + \rho w\vec{k}\right] = 0$$

Or $\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0$ (4)

In case of a steady flow; $\frac{\partial \rho}{\partial t} = 0$

Hence Eq. (4.4) becomes

$\nabla.\left(\rho\vec{V}\right) = 0$ in a rectangelar cartesian system. (5)

$$\frac{\partial}{\partial x}\ (\rho u) + \frac{\partial}{\partial y}\ (\rho v) + \frac{\partial}{\partial z}\ (\rho w) = 0 \qquad (6)$$

Eq. (5&6) represents (C.E) for a steady flow, in case of incompressible flow, ρ= constant

$\nabla.\left(\rho\vec{V}\right) = \rho\nabla.\left(\vec{V}\right)$ is the (C.E) for an incompressible fluid

$$\therefore \rho\ \nabla.\left(\vec{V}\right) = 0 \quad \rightarrow \quad \nabla.\left(\vec{V}\right) = 0 \qquad (7)$$

$$or \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (8)$$

Eq. 4.8 can be written in terms of the strain rate components as

$$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0 \qquad (9)$$

The left side of the Eq's. (8.9) can be physically identified as the rate of volumetric dilatation per unit volume of fluid element in motion is obviously zero for incompressible flow.

Ex.1

The velocity distribution for a three –dimensional incompressible steady state flow is given by

$u = 2x^2 - xy + z^2,\ v = x^2 - 4xy + y^2;\ w = -2xy - yz + y^2$

Show that it satisfies C.E in 3-dimensions.

Sol.

in 3-D C.E is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad since\ \frac{\partial \rho}{\partial t} = 0, \rho = const\ .$$

$$\frac{\partial u}{\partial x} = 4x - y\ ;\ \frac{\partial v}{\partial y} = -4x + 2y\ ;\ \frac{\partial w}{\partial z} = -y$$

Substituting in C.E

$$4x - y + (-4x + 2y) - y = 0$$

$4x - y - 4x + y = 0 \quad statisfing\ C.E\ .$

Ex.2

Derive the continuity equation in cylindrical polar coordinate system Eq. 3.

Sol.

From Fig.7 the rate of mass entering the control volume through face ABCD=$\rho V_r\, rd\theta dz$, and the rate of mass leaving the C.V through the face

$$EFGH = \rho V_r rd\theta\ dz + \frac{\partial}{\partial r}\left(\rho V_r\ rd\theta dz\right)dr$$

Therefore the net rate of mass efflux in the r-direction

$$= \rho V_r\ rd\theta\ dz + \frac{\partial}{\partial r}\left(\rho V_r\ rd\theta\ dz\right)dr - \rho V_r\ rd\theta\ dz$$

$$= \frac{\partial}{\partial r}\left(\rho V_r\ rd\theta\ dz\ dr\right) = \frac{\partial}{\partial r}\left(\rho V_r\right)d\forall\ = \ \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r V_r\right)d\forall$$

$$d\forall = rdr\ d\theta\ dz$$

The net rate of mass efflux from C.V in θ direction = (mass leaving through face ADHE- mass entering through face BCGF)

$$= \rho V_\theta dr\ dz + \frac{\partial}{\partial \theta}\left(\rho V_\theta\ dr\ dz\right)d\theta - \rho V_\theta\ drdz = \frac{1\partial}{r\partial\theta}\left(\rho V_\theta rdrd\theta dz\right) \quad = \frac{1\partial}{r\partial\theta}\left(\rho V_\theta\right)d\forall$$

The net rate of mass efflux in z-direction by similar fashion

$$= \rho V_z\ dr(rd\theta) + \frac{\partial}{\partial z}\left(\rho V_z\, rd\theta\ dr\right)dz - \rho V_z\ dr\ (rd\theta)$$

$$= \frac{\partial}{\partial z}\left(\rho V_z\ rd\theta drdz\right) = \frac{\partial}{\partial z}\left(\rho V_z\right)d\forall$$

The rate of increase of mass within the C.V becomes

$$= \frac{\partial}{\partial z}\left(\rho V_z\right)d\forall + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho V_r\ r\right)d\forall + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho V_\theta\right)d\forall$$

Hence, the fixed form of C.E in a cylindrical polar coordinate system becomes per unit volume

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho V_r\ r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right) = 0$$

$$or \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(\rho V_r\right) + \frac{\rho V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial Z}\left(\rho V_z\right) = 0$$

In case of an incompressible flow.

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

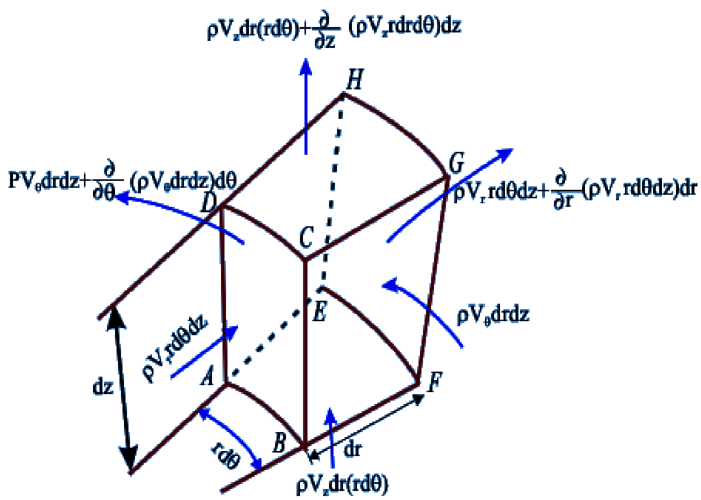


Figure 7: A control volume appropriate to a cylindrical polar coordinate system.

D- Free Body Method.

From Fig. 8 the fluid at line KL moves to new position K'L' in time Δt form the mass conservation law, the mass in (KK') equal to mass in (LL'), then

$\rho_1 A_1\, ds_1 = \rho_2 A_2\, ds_2$ (b1)

Divided the Eq. b1 by Δt

$\rho_1 A_1\, V_1 = \rho_2 A_2\, V_2$ (b2)

(V_1&V_2) represents the average velocity in cross-section *1&2*, A_1&A_1 represents the area of cross-section of the pipe in 1&2. Eq. (b2) represents the mass per time (kg/s)

$\dot{m} = \rho A V = const.$ (b3)

If the fluid is incompressible ρ=const.

$A_1\, V_1 = A_2\, V_2$ (b4)

The rate of flow as discharge, is defined as the quantity of a liquid flowing per second through a section of pipe or a channel and it's denoted by Q.

∴ Discharge, Q= A*V = (m^3/s)

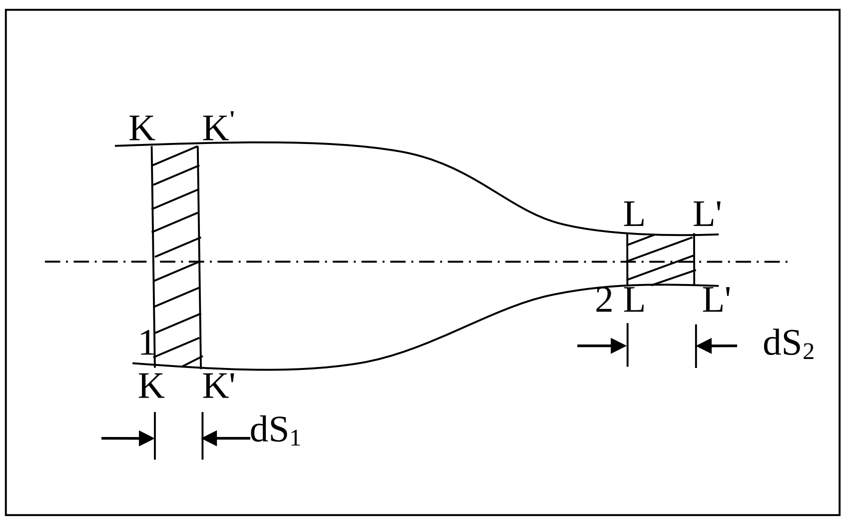


Figure 8: Free body diagram.

Ex-3.

As in Fig.8 the diameter at cross-section (1) is equal to (12 cm), the diameter at cross-section (2) is equal to (8 cm). If the velocity at section (1) is 1.5 m/s, calculate the velocity at section (2)

Sol.

the cross-section area at (1) is

$A_1 = \frac{\pi d_1^2}{4} = \pi \frac{(0.12)^2}{4} = 0.0113\, m^2$

$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi (0.08)^2}{4} = 5.026 * 10^{-3} m^2$

$A_1\, V_1 = A_2\, V_2$

$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.0113*1.5}{5.026*10^{-3}} = 3.375\, m/s$