Lecture Nine

Fluid Motions

1- Streamlines, Path Lines, Stream Tube, Streak Lines.

a- Streamline.

At any instant can be defined as an imaginary curve or line in the flow field so that the tangent to the curve at any point represents the direction of the instantaneous velocity at that point as shown in Fig.1.

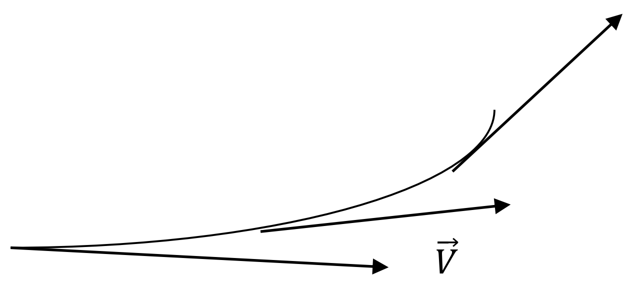


Figure 1: Stream line.

- In an unsteady flow where velocity vector change with time the pattern of stream lines also changes from instant to instant
- In a steady flow, the orientation as the pattern of stream line will be fixed

from above definition of stream line it can be written as

$\vec{V} \times d\vec{s} = 0$

$d\vec{s}$ The length of an infinitesimal line segment along a stream line at a point.

$\vec{V}$ The instantaneous velocity vector.

$d\vec{s} = \vec{i}dx + \vec{j}dy + \vec{k}dz \quad ; \qquad \vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$

$\vec{V} \times d\vec{s} = 0$

Or

$$\begin{vmatrix} i & j & k \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0 \quad --\rightarrow udy = vdx; \;\; udz = wdx \,; \quad vdz = wdy$$

Or $\quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

Ex.1

Determine the stream lines for the two-dimensional steady flow if the velocity field is given by

$$\vec{V} = \left(\frac{V_0}{l}\right)(x\vec{i} - y\vec{j})$$

V_0 & l are constant.

Sol.

$u = \left(\frac{V_0}{l}\right)x \,; v = -\left(\frac{V_0}{l}\right)y$

$\frac{dy}{dx} = \frac{v}{u} = -\frac{\left(\frac{V_0}{l}\right)y}{\left(\frac{V_0}{l}\right)x} = -\frac{y}{x}$

Or by integration

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

Or $lny = -lnx + constant.$

Along stream line ***xy* = *C*** ; C is constant

Different value of C, we can plot various lines in ***x*-*y*** plane.

Following points about stream lines are worth noting

1- A stream line cannot intersect itself nor two stream lines can cross.
2- There cannot be any movement of the fluid mass across the streamlines
3- Streamline spacing varies inversely as the velocity; converging of stream lines in any particular direction shows accelerated flow in that direction.
4- Whereas a path lines gives the path of one particular particle at successive instant of time a streamline indicates the direction of a number of particles at the same instant.
5- The series of streamlines represent the flow pattern at an instant as in Fig. 2.

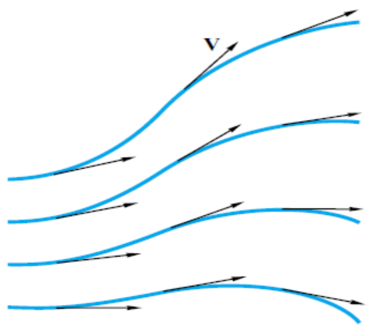


Figure 2: Series of streamlines.

Ex.2

Obtain the equation to the streamlines for the velocity field given as:

$$\vec{V} = 2x^3\vec{i} - 6x^2y\vec{j}$$

Sol.

$$u = 2x^3; \ v = -6x^2y$$

The stream line in two dimensions are defined by

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{6x^2y}{2x^3} = -\frac{3y}{x}$$

Separating the variables; we have $\frac{dy}{y} = -\frac{3dx}{x}$ By integration

$$\log_e y = -3 \ \log_e x + c1$$

Or

$$\log_e y + 3 \ \log_e x = c1 \qquad yx^3 = c$$

Ex.3

For a three-dimensional flow the velocity distribution is given by u = -x, v= 3-y and w = 3-z, what is the equation of a stream line passing through (1,2,2)?

Sol.

The streamlines are defined by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Substituting for u, v & w we get

$$\frac{dx}{-x} = \frac{dy}{3-y} = \frac{dz}{3-z}$$

Take the first two term; by integration

$$\int \frac{dx}{-x} = \int \frac{dy}{(3-y)}$$

$$-\log_e x = -\log_e (3-y) + c1$$

Where c1= constant of integration

Since the streamline passes through x=1, y=2 ∴ c1=0

$(x)^{-1} = (3-y)^{-1} \; or \; x = (3-y)$

$$and \int \frac{dx}{-x} = \int \frac{dz}{3-z}$$

$$-\log_e x = -\log_e (3-z) + c2$$

$$at\ x = 1\,, z = 2 \qquad \therefore c2 = 0$$

$$x^{-1} = (3-z)^{-1} \qquad --\rightarrow x = (3-z)$$

$$x = (3-y) = (3-z)$$

b- Path Lines.

A path line is the trajectory of fluid particle of fixed identity or a path line shows the direction of particular particle as it moves ahead as shown in Fig.3.

- *One dimension flow* **:-** the single space coordinate is usually and time as flow in pipe the average values of the flow parameters are assumed
- *Two dimension flow***:-** All the flow parameters are functions of time & 2-space coordinates say (x& y)
- *Three dimensional flow***:-** the parameters are function of three space coordinates and time.

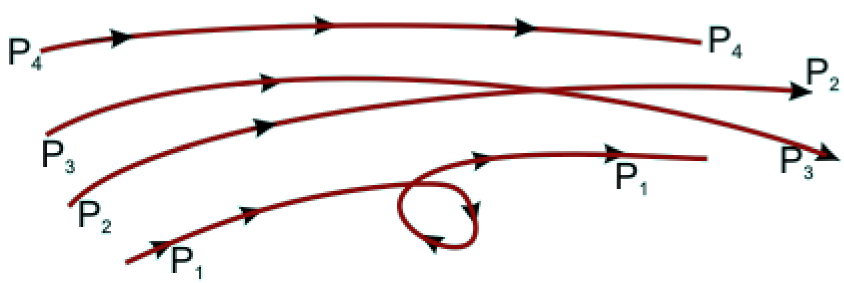


Figure 3: Series of path lines.

c- Stream Tube.

A fluid mass bounded by a group of stream lines. The contents of a stream tube are known as "current filament". Example, the flow as in pipes and nozzles

1- The stream tube has finite dimensions
2- As there is no flow perpendicular to stream lines therefore, there is no flow across the surface (called stream surface) as shown in Fig.4.

3- The shape of a stream tube change from one instant to another, because of change is the position of streamlines.

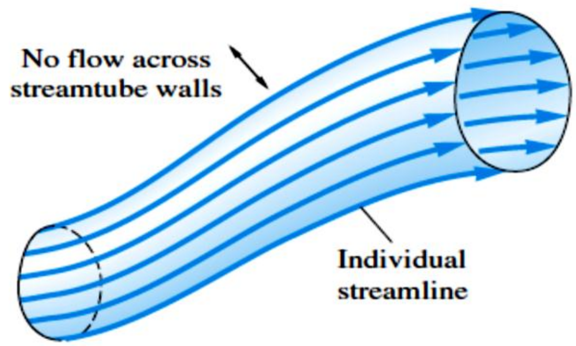


Figure 4: Stream tube is formed by closed collection of streamlines.

d- Streak Lines.

Streak lines provide an instantaneous picture of the particles, which have passed through a given point like the injection point of a dye in a flow. In steady flow these lines will be coincide with stream lines. Fig. 5 shows the path line and streak line. Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3, and 4. A line joining these points is the streak line.

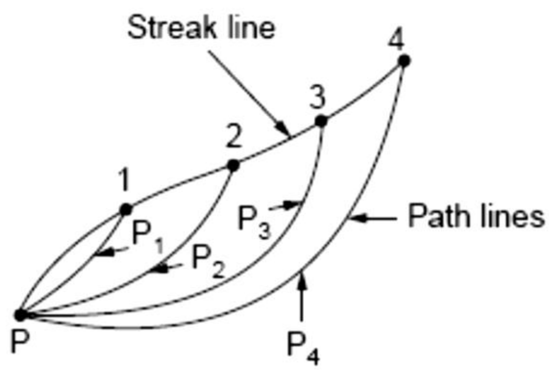
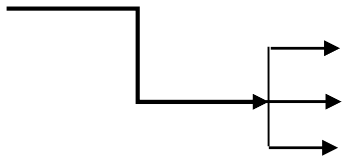


Figure 5: Path lines and streak lines.

2- Movement of Fluid Element.

The movement of fluid element has three distinct features in space simultaneously.



Translation
Rate of deformation
Rotation

A- Pure Translation.

In Fig 6, the fluid element in pure translation this occur in the uniform flow field. **In absence of deformation and rotation**,

a) There will be no change in the length of the sides of the fluid element.
b) There will be no change in the included angles made by the sides of the fluid element.
c) The sides are displaced in parallel direction.

This is possible when the flow velocities u (the x component velocity) and v (the y component velocity) are neither a function of x nor of y, i.e., the flow field is totally uniform.

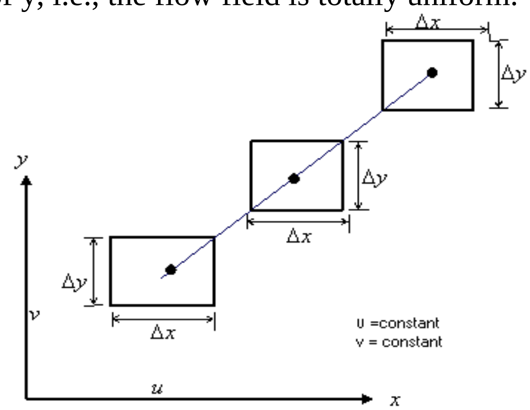


Figure 6: Fluid element in pure translation.

B- Linear Deformation.

If a component of flow velocity becomes the function of only one space coordinate along which that velocity component is defined. For example,

- if $u = u(x)$ and $v = v(y)$, the fluid element ABCD suffers a change in its linear dimensions along with translation
- there is no change in the included angle by the sides as shown in Fig.7

The relative displacement of point B with respect to point A per unit time in x direction is $\frac{\partial u}{\partial x}\Delta x$

Similarly, the relative displacement of D with respect to A per unit time in y direction is $\frac{\partial v}{\partial y}\Delta y$

Hence, the sides move parallel from this initial position and without changing the included angle. This situation is referred to as translation with linear deformation.

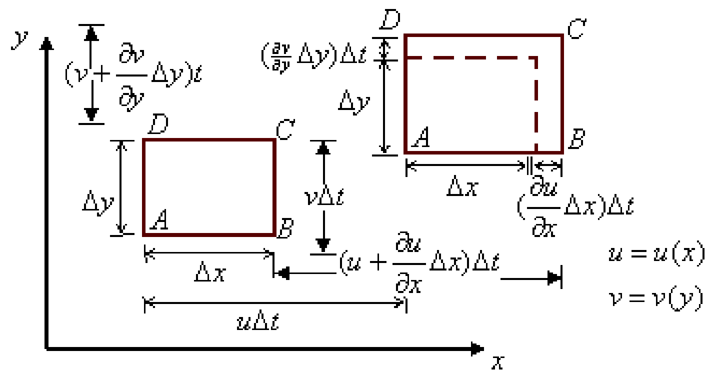


Figure 7: Fluid element in translation with continuous linear deformation.

Observations from Fig. 7 gives:

- Since u is not a function of y and *v* is not a function of x.
- All points on the linear element AD move with same velocity in the x direction.
- All points on the linear element AB move with the same velocity in y direction.
- Hence the sides move parallel from their initial position without changing the included angle.
  This situation is referred to as translation with linear deformation.

Strain rate: - The changes in lengths along the coordinate axes per unit time per unit original lengths are defined as the **components of linear deformation or strain rate in the respective directions.**

$\in_{xx}=\frac{\partial u}{\partial x}$ Linear strain rate component in the x direction.

$\in_{yy}=\frac{\partial v}{\partial y}$ Linear strain rate component in y direction.

C- Rate of Deformation in the Fluid Element.

Let us consider both the velocity component ***u*** and ***v*** are functions of ***x*** and ***y***, i.e., *u = u(x,y)* & *v = v(x,y)*. Fig. 8 represents the above conditions,
observations from the figure:

- Point B has a relative displacement in y direction with respect to the point A.
- Point D has a relative displacement in x direction with respect to point A.
- The included angle between AB and AD changes.
- The fluid element suffers a continuous angular deformation along with the linear deformations in course of its motion.

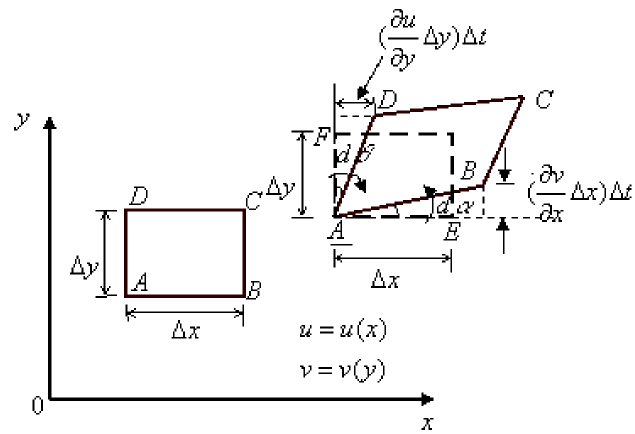


Figure 8: Fluid element in translation with simultaneous linear and angular deformation rates.

Rate of Angular deformation: - is defined as the rate of change of angle between the linear segments AB & AD which were initially perpendicular to each other.
The rate of angular deformation is
$\dot{\gamma}_{xy} = \left(\frac{d\propto}{dt} + \frac{d\beta}{dt}\right)$ From geometry
Hence $\frac{d\propto}{dt} + \frac{d\beta}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$
Finally
$\dot{\gamma}_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

D- Rotation.

The transverse displacement of B with respect to A & lateral displacement of D with respect to A as in Fig. 3.8 is called the rotation of AB & AD about A. The rotation at a point is defined as the arithmetic mean of the angular velocity of two perpendicular linear segments meeting at that point. The angular velocities of AB & AD about A are $\frac{d\propto}{dt}$ & $\frac{d\beta}{dt}$ $\quad repectively$
Considering the anticlockwise direction is positive, the rotation at A can be written as,

$$\omega_z = \frac{1}{2}\left(\frac{d\propto}{dt} - \frac{d\beta}{dt}\right) \qquad (9)$$

From Fig. 9, $d\alpha$ and $d\beta$ are each directly related to velocity derivatives in the limit of small *dt*

$$d\alpha = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial v/\partial x)dxdt}{dx+\left((\partial u/\partial x)dxdt\right)}\right] = \frac{\partial v}{\partial x}dt \qquad (10)$$

$$d\beta = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial u/\partial y)dydt}{dy+\left((\partial v/\partial y)dydt\right)}\right] = \frac{\partial u}{\partial y}dt \qquad (11)$$

Combining Eq's. (10 and 11) with Eq. 9 obtain the following

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (12)$$

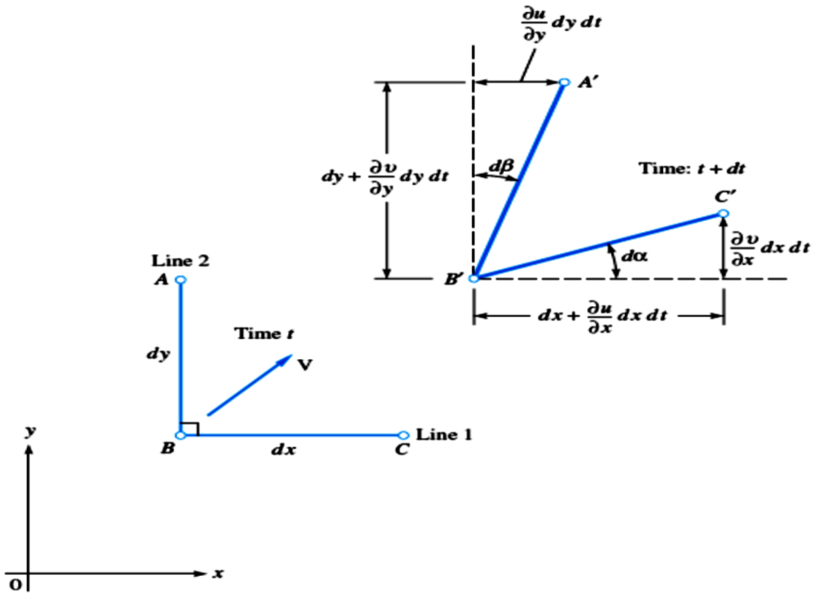


Figure 9: Two fluid lines deforming in xy plane.

The suffix ***z*** in ω represents the rotation about z-axis in the case of two dimensional flow along (x and y).

Rotation of $\vec{V}$, written $\frac{1}{2}(\nabla \times \vec{V})$ is curl $\vec{V}$ or rot $\vec{V}$ is defined by

$$\vec{\omega} = \frac{1}{2}\left(\nabla \times \vec{V}\right) = \frac{1}{2}\left[\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(u\vec{i} + v\vec{j} + w\vec{k}\right)\right]$$

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v & w \end{vmatrix}\vec{i} - \frac{1}{2}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ u & w \end{vmatrix}\vec{j} + \frac{1}{2}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & v \end{vmatrix}\vec{k} \qquad (13)$$

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$\vec{\omega} = \frac{1}{2}\left(\omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}\right) \qquad (14)$$

For three-dimensional flow the rotation is possible about three-axes. The expression for rotation ω_x, ω_y and ω_z can be obtained in like manner,

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad (15)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (16)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (17)$$

In the vector notation, the above equation can be rewritten as

$$\vec{\omega} = \frac{1}{2}\left(\omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}\right) = \frac{1}{2}\left(\nabla \times \vec{V}\right) \qquad (18)$$

The vector $\left(\nabla \times \vec{V}\right)$ is the curl of velocity vector. The motion is described as irrotational when the components of rotation are zero.

For irrotational flow, the angle of rotation of the axes towards each other or away from each other should be equal *i.e.*, the condition to be satisfied for irrotational flow is,

$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ Or $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ for irrotational flow. (19)

Ex.4

Verify whether the following flow fields are rotational. If so, determine the components of rotation about the coordinate axes,

$(i)\quad u = xyz,\ v = xz,\ w = \frac{1}{2}yz^2 - xy$

$(ii)\quad u = xy,\ v = \frac{1}{2}(x^2 - y^2)$

Sol. *(i)*

$\frac{\partial u}{\partial y} = xz;\quad \frac{\partial v}{\partial x} = z;\quad \frac{\partial w}{\partial x} = -y;\quad \frac{\partial w}{\partial y} = \frac{1}{2}z^2 - x$

$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(z - xz) = \frac{z}{2}(1 - x)$

$\omega_x = \frac{1}{2}\left(\frac{\partial w}{dy} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left[\left(\frac{1}{2}z^2 - x\right) - x\right] = \frac{1}{2}\left(\frac{1}{2}z^2 - 2x\right)$

$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(yx + y) = \frac{y}{2}(x + 1)$

$(ii)\ u = xy;\ v = \frac{1}{2}(x^2 - y^2);\ \frac{\partial u}{\partial y} = x;\ \frac{\partial v}{\partial x} = \frac{2}{2}x = x$

$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(x - x) = 0$ Irrotational Flow

Ex.5

Given that

$u = -4ax(x^2 - 3y^2)$

$v = 4ay(3x^2 - y^2)$

Examine whether these velocity components represent a physically possible two-dimensional flow, if so whether the flow is rotational or irrotational?

Sol.

Given u ,v is x, y components

$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[-4ax(x^2 - 3y^2)] = \frac{\partial}{\partial y}(-4ax^3 + 12axy^2) = 24axy$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}[4ay(3x^2 - y^2)] = \frac{\partial}{\partial x}(12ayx^2 - 4ay^3] = 24ayx$

$\omega_z = \frac{1}{2}(24ayx - 24ayx) = 0, henc\ the\ flow\ is\ irrotational.$

Ex.6

Examine whether the flow is rotational or irrotational at the point (1,-1,1) for the following velocity field $\left(\vec{V} = xz^3\vec{\imath} - 2x^2yz\vec{\jmath} + 2yz^4\vec{k}\right)$

Sol.

$\vec{\omega} = \frac{1}{2}\left(\omega_x\vec{\imath} + \omega_y\vec{\jmath} + \omega_z\vec{k}\right) = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right]$

$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2xy^2z)\right]\vec{\imath} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\vec{\jmath} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\vec{k}$

$\vec{\omega} = (z^4 + (x^2y)\vec{\imath} + \left(\frac{3}{2}xz^2 - 0\right)\vec{\jmath} + (-2xyz - 0)\vec{k}$

$$\vec{\omega} = \frac{3}{2}\vec{\jmath} + 2\vec{k} \qquad at\ (1, -1, 1)$$