Lecture Eight

Fluid Flow Concept

1- *Definitions.*

A. *Kinematics of Fluid.* Is the geometry of motion, which is describes the fluid motion and its consequences without consideration of the nature of forces causing the motion.
The subject has three main aspects

Kinematic of Fluid →
- [Development of methods & techniques for describing and specifying the motion of fluid]
- [Determination of the conditions for the kinematic possibility of fluid motion]
- [Characterization of different types of motion and associated deformation rates of any fluid element]

B. *Ideal Fluid.* Is a frictionless and incompressible fluid, the flow processes are reversible and nonviscous.

C. *Laminar Flow.* When the fluid particles move along smooth paths in laminas, or layers with one layer gliding smoothly over an adjacent layer this flow is called laminar.

D. *Turbulent Flow.* The fluid particles are moving in very irregular paths causing an exchange of momentum from one portion of the fluid to another.

E. *Scalar & vector fields.*

Scalar: - scalar is a quantity which can be expressed by a single number representing its magnitude, as mass, density, and temperature.

Scalar field: - If at every point in a region, a scalar function has a defined value, the region is called a scalar field, as temperature distribution in a rod.

Vector:- Vector is a quantity which is specified by magnitude and direction, as force, velocity and displacement.

Vector field: - If at every point in a region, a vector function has a defined value, the region is called a vector field, as, velocity field of a flowing fluid.

F. *Flow field:* - The region in which the flow parameters, as velocity, pressure etc, are defined at each and every point at any instant of time is called a flow field.

2- *Description of Fluid Motion.*

a. Lagrangian Method (L.M.).

The fluid motion is described by tracing the kinematic behavior of each particle constituting the flow. This method depends on the identities of the particles are made by specifying their initial position (spatial location) at a given time. The position of a particle at any other instant of time then becomes a function of its identity and time.
Analyticaly can be expressed as

$\vec{S} = S(\vec{S}_0, t)$

$\vec{S}$ is the position vector of a particle with respect to a fixed point of reference at a time (t).

$\overrightarrow{S_0}$ Its initial position at a given time $t=t_0$

The above equation can be written into scalar components with respect to a rectangular cartesian frame of coordinates as:

$X=X(x_0,y_0,z_0,t)$ (1.a)

$Y=Y(x_0,y_0,z_0,t)$ (1.b)

$Z=Z(x_0,y_0,z_0,t)$ (1.c)

Where, x_0, y_0, z_0 are the initial coordinates and x, y, z are the coordinates at time t of the particles

Hence, $\vec{S}$ can be expressed as

$\vec{S}= \vec{\imath}x + \vec{\jmath}y + \vec{k}z$ where $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ are the unit vectors along x ,y and z axes respectively .

The velocity $\vec{V}$ and acceleration $\vec{a}$ of the fluid particle can be obtained from the material derivatives of the position of the particle with respect to time. Therefore,

$$\vec{V} = \left[\frac{ds}{dt}\right]_{s0} \quad (2)$$

In terms of scalar components

$$u = \left(\frac{dx}{dt}\right)_{x0,y0,z0} \quad (2.a)$$

$$v = \left(\frac{dy}{dt}\right)_{x0,y0,z0} \quad (2.b)$$

$$w = \left(\frac{dz}{dt}\right)_{x0,y0,z0} \quad (2.c)$$

Where u , v , w are the components of velocity in x , y, z direction respectively.

Similarly, for the acceleration

$$\vec{a} = \left[\frac{d^2\vec{S}}{dt^2}\right] \quad (3)$$

Hence ,

$$a_x = \left[\frac{d^2x}{dt^2}\right]_{x0,y0,z0} \quad (3.a)$$

$$a_y = \left[\frac{d^2y}{dt^2}\right]_{x0,y0,z0} \quad (3.b)$$

$$a_z = \left[\frac{d^2z}{dt^2}\right]_{x0,y0,z0} \quad (3.c)$$

Where a_x , a_y, a_z are the acceleration in x , y and z direction respectively.

Advantage of L.M

1- The motion & trajectory of each fluid particle is known.

2- The particles are identified at the start and traced throughout their motion.

Disadvantage: the solution of the equations presents appreciable mathematical difficulties.

b. Eulerian Method.

It avoids the determination of the movement of each individual fluid particle in all details. It seeks the velocity $\vec{V}$ and its variation with time t at each and every location $\vec{s}$ in the flow field. Mathematical representation of the flow field in Eulerian method

$$\vec{V} = V(\vec{S}, t) \quad (4)$$

where

$$\vec{V} = \vec{\imath}u + \vec{\jmath}v + \vec{k}w \quad and \quad \vec{S} = \vec{\imath}x + \vec{\jmath}y + \vec{k}z$$

Therefor

u=u(x ,y ,z ,t)
v=v(x ,y ,z ,t)
w=w(x ,y ,z ,t)

3- *Variation of Flow Parameters in Time & Space.*

Hydrodynamic parameters like pressure and density along with flow velocity may vary from one point to another and also from one instant to another at a fixed point. According to type of variation

A- ***Steady flow***. A steady flow is defined as a flow in which the various hydrodynamic parameters and fluid properties at any point do not change with time.

$$\partial\rho/_{\partial t} = 0, \quad \partial p/_{\partial t} = 0, \quad \partial T/_{\partial t} = 0$$

In Eulerian approach a steady flow is described as

$\vec{V} = V(\vec{S})$And $\vec{a} = a(\vec{S})$

The hydrodynamic parameters may vary with location, but the spatial distributions of these parameters remain invariant with time. In Lagrangian approach, the velocities of all points passing through any fixed point at different times will be same. Therefore, the Eulerian and Lagrangian approach of describing fluid motion become identical under this situation.

B- ***Unsteady flow.*** Is defined as a flow in which the hydrodynamic parameters and fluid properties changes with time.

$$\partial\rho/_{\partial t} \neq 0, \quad \partial p/_{\partial t} \neq 0, \ \partial T/_{\partial t} \neq 0$$

C- ***Uniform flow.*** the flow is defined as uniform flow when in the flow field the velocity and other hydrodynamic parameters do not change from point to point at any instant of time can be expressed as : v=v(t)

Any hydrodynamic parameter will have one value in the entire field
If changes with time → unsteady uniform flow
OR
Does not change with time → steady uniform flow

D. ***Non–uniform flow.*** When the velocity other hydrodynamic parameters changes from one point to another the flow is defined as non- uniform.

Non- uniform may be found either in the direction of flow or in direction perpendicular it.

4- *Material Derivative and Acceleration.*

The velocity components u,v,w of the particle along x, y and z direction in space , can be written in Eulerian form as

u=u (x , y, z)
v=v(x , y, z)
z=z(x , y, z)

After an infinitesimal time interval t, let the particle move to a new position given by the coordinates (x+Δx,y+Δy,z+Δz)

Its velocity components at this new position be (u+Δu ,v+Δv, w+Δw).

Expression of velocity components in the Taylor's series form

$$u + \Delta u = u(x, y, z, t) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t +$$
$$higher\ order\ terms\ \ in\ \Delta x, \Delta y, \Delta z \& \Delta t.$$

$$v + \Delta v = v(x, y, z, t) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\,\Delta y + \frac{dv}{dz}\,\Delta z + \frac{\partial v}{\partial t}\,\Delta t + \ higher\ order\ termsin\ \Delta x, \Delta y, \Delta z \& \Delta t.$$

$$w + \Delta w = w(x, y, z, t) + \frac{\partial w}{\partial x}\,\Delta x + \frac{\partial w}{\partial y}\,\Delta y + \frac{\partial w}{\partial z}\Delta z + \frac{\partial w}{\partial t}\,\Delta t + higher\ order\ terms\ \ in\ \Delta x, \Delta y, \Delta z \& \Delta t.$$

The increment in space coordinates can be written as:-

$\Delta x = u\Delta t,\ \Delta y = v\Delta t, and\ \Delta z = w\Delta t$

Substituting the value of Δx, Δy, Δz in above eqn. , we have

$$\frac{\Delta u}{\Delta t} = u\frac{\partial u}{\partial x} + \ v\frac{\partial u}{\partial y} + \ w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\ , etc$$

In the limit Δt→ 0 , the equation becomes

$$\left.\begin{aligned} \frac{Du}{Dt} &= \frac{\partial u}{\partial t} + \ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \frac{Dv}{Dt} &= \frac{\partial v}{\partial t} + \ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \ w\frac{\partial v}{\partial z} \\ \frac{Dw}{Dt} &= \frac{\partial w}{\partial t} + \ u\frac{\partial w}{\partial x} + \ v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \qquad (5)$$

The above Eq's. tell that the operator for total differential with respect to time, D/Dt in a convective field is related to the partial differential as:

$$\underbrace{\frac{D}{Dt}}_{\text{Material or substantial derivative}} = \underbrace{\frac{\partial}{\partial t}}_{\text{Temporal or local derivative}} + \underbrace{u\frac{\partial}{\partial x} + \ v\frac{\partial}{\partial y} + \ w\frac{\partial}{\partial z}}_{\text{convective derivative}} \qquad (6)$$

From Eq. 5

- The terms in the left hand sides in (x,y,z) are the component of substantial or material acceleration.
- The first terms in the *R.H.S* are the local or temporal accelerations
- While the other terms are convective accelerations.

Thus we can write,

$$\begin{aligned} a_x &= \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \ u\frac{\partial u}{\partial x} + \ v\frac{\partial u}{\partial y} + \ w\frac{\partial u}{\partial z} \\ a_y &= \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \ u\frac{\partial v}{\partial x} + \ v\frac{\partial v}{\partial y} + \ w\frac{\partial v}{\partial z} \\ a_z &= \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + \ u\frac{\partial w}{\partial x} + \ v\frac{\partial w}{\partial y} + \ w\frac{\partial w}{\partial z} \end{aligned} \qquad (7)$$

Material or substantial acceleration = temporal acceleration + convective acceleration, the total acceleration vector is

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad (8)$$

Ex.1

Given the velocity field

$\vec{V} = (4 + xy + 2t)\vec{i} + 6x^3\vec{j} + (3xt^2 + \ z)\vec{k}$

Find the acceleration of fluid particles

a- As function of x ,y , z and t

b- At (1,1,1) and time t=1sec.

Sol.

a- From the given velocity field

$u = 4 + xy + 2t\,,\;\; v = 6x^3,\;\; w = 3xt^2 + z$

From Eq. 7

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial t} = 2\,;\; \frac{\partial u}{\partial x} = y\,;\; \frac{\partial u}{\partial y} = x\,;\; \frac{\partial u}{\partial z} = o$$

$$a_x = 2 + (4 + xy + 2t)(y) + (6x^3)(x) + (3xt^2 + z)(0)$$

$$a_x = 2 + 4y + xy^2 + 2ty + 6x^4 \qquad (a)$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{\partial v}{\partial t} = 0\,;\frac{\partial v}{\partial x} = 18x^2;\frac{\partial v}{\partial y} = 0;\frac{\partial v}{\partial z} = 0$$

$$a_y = 0 + (4 + xy + 2t)(18x^2) + (3xt^2 + z)(0)$$

$$a_y = 72x^2 + 18yx^3 + 36tx^2 \qquad (b)$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\frac{\partial w}{\partial t} = 6xt;\quad \frac{\partial w}{\partial x} = 3t^2\,;\quad \frac{\partial w}{wy} = 0;\quad \frac{\partial w}{\partial z} = 1$$

$$a_z = 6xt + (\,4 + xy + 2t)(3t^2) + (6x^3)(0) + (3xt^2 + z)(1)$$

$$a_z = 6xt + 12t^2 + 3xyt^2 + 6t^3 + 3xt^2 + z \qquad (c)$$

Combining Eq's (a , b &c) the total acceleration as in Eq. 8

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{a} = (2 + 4y + xy^2 + 2ty + 6x^4)\vec{i} + (72x^2 + 18x^3y + 36tx^2)\vec{j} + (16xt + 12t^2 + 3xyt^2 + 6t^3 + 3xt^2 + z)\vec{k}$$

b-

At 1,1,1 and t=1 acceleration vector is

$$\vec{a} = (2 + 4 + 1 + 2 + 6)\vec{i} + (72 + 18 + 36)\vec{j} + (6 + 12 + 3 + 6 + 3 + 1)\vec{k}$$

$$\vec{a} = 15\vec{i} + 126\vec{j} + 31\vec{k}$$

Ex.2

In a fluid flow, the velocity field is given by

$$\vec{V} = (3x + 2y)\vec{i} + (2z + 3x^2)\vec{j} + (2t - 3z)\vec{k}$$

Determine

a) The velocity components u , v ,w at any point in the flow field
b) The speed at point (1 , 1 , 1)
c) The speed at time t= 2s at point (0,0,2)
d) Classify the velocity field as steady or unsteady, uniform or non uniform and one two or three dimensional.

Sol.

From the given velocity field

a) Velocity components are :

$u = 3x + 2y;\quad v = (2z + 3x^2)\,;\; w = (2t - 3z)$

b) Speed at point (1,1,1); $\vec{V}_{(1,1,1)}$

Substituting x=1, y=1, z=1 in the expression for u, v & w.

$u = (3 + 2) = 5\,, v = (2 + 3) = 5\;\;, w = (2t - 3)$

$$V^2 = u^2 + v^2 + w^2 = 5^2 + 5^2 + (2t - 3)^2 = 25 + 25 + 4t^2 - 12t + 9$$

$V^2 = 4t^2 - 12t + 59 \;\; \rightarrow \;\; V = \sqrt{4t^2 - 12t + 59}$

c) Speed at t= 2s at point (0,0,2):

Substituting t=2, x=0, y=0, z=2 in the expression of u, v & w we get,

$$u = 0\,, \quad v = (2*2) = 4, \quad w = (2*2 - 3*2) = -2$$
$$\mathrm{V}^2 = \mathrm{u}^2 + \mathrm{v}^2 + \mathrm{w}^2 = 0 + 4^2 * (-2)^2 = 20$$
$$\text{Or } \mathrm{V}_{(0,0,2)} = \sqrt{20} = 4.472 \; units$$

Velocity field types

i) Since $\vec{V}$ at given (x,y,z) depends on t, it's unsteady flow
ii) Since at given t velocity changes in x direction it's non-uniform.
iii) Since $\vec{V}$depends on x, y, z; its three dimension flow.

Ex.3

Velocity for a two dimensional flow field is given by

$\vec{V} = (3 + 2xy + 4t^2)\vec{\imath} + (xy^2 + 3t)\vec{\jmath}$

Find the velocity and acceleration at a point (1, 2) after 2s.

Sol.

Velocity $\vec{V}_{(1,2)}$

Substituting x=1, y=2 and t=2 in the expression of velocity field, we get

$\vec{V} = (3 + 2*1*2 + 4*2^2)\vec{\imath} + (1*2^2 + 3*2)\vec{\jmath} = 23\vec{\imath} + 10\vec{\jmath}$

$\therefore \vec{V}_{(1,2)} = \sqrt{23^2 + 10^2} = 25.08\, units.$

Acceleration at point (1,2), $a_{(1,2)}$

We know that $\quad \vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + \left(u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y}\right)$

$\frac{\partial\vec{V}}{\partial x} = 2y\vec{\imath} + y^2\vec{\jmath}$

$\frac{\partial\vec{V}}{\partial y} = 2x\vec{\imath} + 2xy\vec{\jmath}$

$\frac{\partial\vec{V}}{\partial t} = 8t\vec{\imath} + 3\vec{\jmath}$

$$\vec{a} = (3 + 2xy + 4t^2)(2y\vec{\imath} + y^2\vec{\jmath}) + (xy^2 + 3t)(2x\vec{\imath} + 2xy\vec{\jmath}) + (8t\vec{\imath} + 3\vec{\jmath})$$
$$\vec{a} = (3 + 2*1*2 + 4*2^2)(2*2\vec{\imath} + 2^2\vec{\jmath}) + (1*2^2 + 3*2)(2*1\vec{\imath} + 2*1*2\vec{\jmath}) + (8*2\vec{\imath} + 3\vec{\jmath})$$

$\vec{a} = 23(4\vec{\imath} + 4\vec{\jmath}) + 10(2\vec{\imath} + 4\vec{\jmath}) + (16\vec{\imath} + 3\vec{\jmath})$

$\vec{a} = 92\vec{\imath} + 92\vec{\jmath} + 20\vec{\imath} + 40\vec{\jmath} + 16\vec{\imath} + 3\vec{\jmath} = 128\vec{\imath} + 135\vec{\jmath}$

$a_{1,2} = \sqrt{128^2 + 135^2} = 186.03\, units$

Ex.4

Find the velocity and acceleration at a point (1, 2, 3) after 1s for a three-dimensional flow given by

$$u = yz + t, \qquad v = xz - t, \qquad w = xy$$

Sol.

Given; three-dimensional flow field velocity at a point 1,2,3 V(1,2,3) after 1 sec. is

$$\mathrm{u} = \mathrm{yz} + \mathrm{t} = 2*3 + 1 = 7\frac{\mathrm{m}}{\mathrm{s}}$$
$$\mathrm{v} = \mathrm{xz} - \mathrm{t} = 1*3 - 1 = 2\frac{\mathrm{m}}{\mathrm{s}}$$

$$w = xy = 1 * 2 = 2\frac{m}{s}$$

$\vec{V}_{(1,2,3)} = 7\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$

$V = \sqrt{7^2 + 2^2 + 2^2} = 7.55\frac{m}{s}$

Acceleration $\vec{a}_{(1,2,3)}$

Now $V = (yz + t)\vec{\imath} + (xz - t)\vec{\jmath} + xy\vec{k}$

Acceleration

$\vec{a} = \frac{d\vec{V}}{dt} = \left(u * \frac{\partial\vec{V}}{\partial x} + v * \frac{\partial\vec{V}}{\partial y} + w * \frac{\partial\vec{V}}{\partial z}\right) + \frac{\partial\vec{V}}{\partial t}$

$\vec{a} = (yz + t) * \left(z\vec{\jmath} + y\vec{k}\right) + (xz - t)\left(z\vec{\imath} + x\vec{k}\right) + xy(y\vec{\imath} + x\vec{\jmath}) + (1\vec{\imath} - 1\vec{\jmath})$

$\vec{a}_{(1,2,3)} = 7\left(3\vec{\jmath} + 2\vec{k}\right) + 2\left(3\vec{\imath} + 1\vec{k}\right) + 2(2\vec{\imath} + 1\vec{\jmath}) + (1\vec{\imath} - 1\vec{\jmath})$

$\vec{a}_{(1,2,3)} = \left(21\vec{\jmath} + 14\vec{k} + 6\vec{\imath} + 2\vec{k} + 4\vec{\imath} + 2\vec{\jmath}\right) + (1\vec{\imath} - 1\vec{\jmath})$

$\vec{a}_{(1,2,3)} = \left(10\vec{\imath} + 23\vec{\jmath} + 16\vec{k}\right) + (1\vec{\imath} - 1\vec{\jmath})$

The convective acceleration component are :(10, 23, 16) m/s^2

The local acceleration components are: $(1, -1)\frac{m}{s^2}$ along x and y directions

The total acceleration of fluid particles at the point (1, 2, 3) is

$a_{1,2,3} = \sqrt{(10+1)^2 + [23 + (-1)]^2 + 16^2} = \sqrt{11^2 + 22^2 + 16^2} = 29.34\frac{m}{s^2}$