Numrical Integration: applications using Newton-Cotes Integration Formulas

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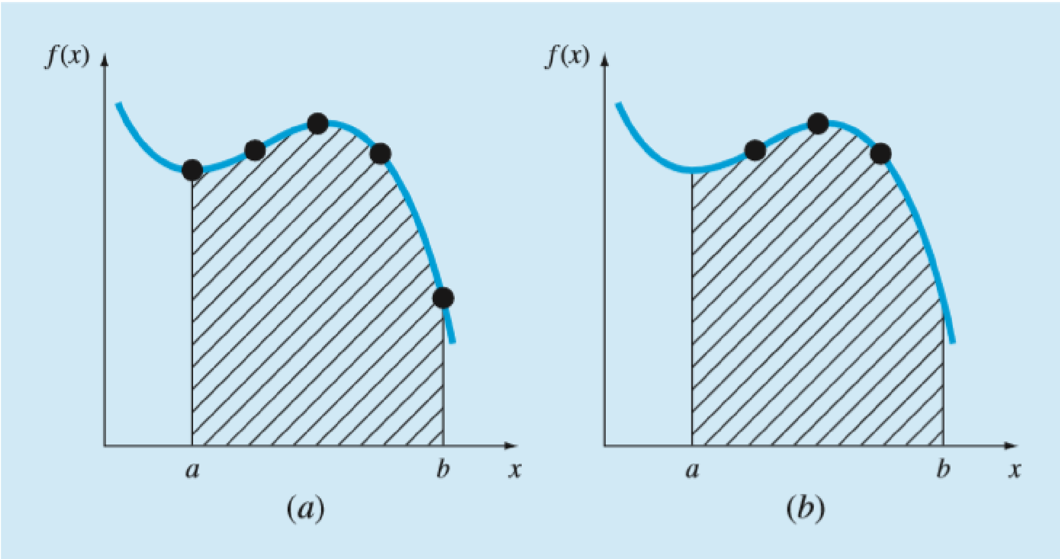
The Newton-Cotes formulas are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate.

Closed and open forms of the Newton-Cotes formulas are available. The closed forms are those where the data points at the beginning and end of the limits of integration are known. The open forms have integration limits that extend beyond the range of the data.

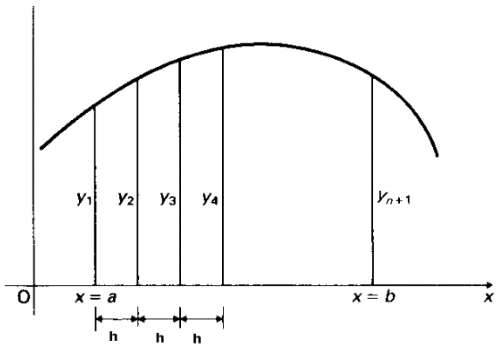


The difference between (a) closed and (b) open integration formulas.

<u>Closed forum:</u>

1- Trapezium Rule

Finding a definite integral can be thought of as determining the area under the curve. Some integrals are difficult to evaluate exactly and so numerical methods are needed.



The simplest of these methods is the trapezium rule which approximates the area under the curve by *n* trapezia each of width *h* as shown.
The formula for the area of the first of these trapezia is

$$A=\frac{h}{2}(y_0+y_1)$$

when we sum all these areas, we will get

$$\int_a^b y\,\mathrm{d}x \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right]$$

where $h = \dfrac{b-a}{n}$

Example

1(i) Consider $I = \int_1^2 x^2\,\mathrm{d}x = \left[\dfrac{x^3}{3}\right]_1^2 = \dfrac{2^3}{3} - \dfrac{1}{3} = \dfrac{7}{3}$ $= 2.\dot{3}$

(ii) We have, using trapezium rule with 1 interval $I \approx \dfrac{h}{2}\left(y_0 + y_1\right)$ $h = \dfrac{2-1}{1} = 1$

	x	$y = x^2$
y_0	1	1^2
y_1	2	2^2

$I \approx \dfrac{1}{2}\left(1^2 + 2^2\right) = \dfrac{1}{2} \times 5$

$= 2.5$

(iii) Using 2 intervals: $I \approx \frac{h}{2}(y_0 + 2y_1 + y_2)$ $\qquad h = \frac{2-1}{2} = \frac{1}{2}$

	x	$y = x^2$
y_0	1	1^2
y_1	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^2$
y_2	2	2^2

$$I \approx \frac{1}{4}\left(1^2 + 2\left(\frac{3}{2}\right)^2 + 2^2\right)$$

$$= \frac{1}{4}\left(1 + \frac{9}{2} + 4\right)$$

$$= \frac{19}{8}$$

$$= 2.375$$

(iv) Using 4 intervals: $I \approx \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3) + y_4)$ $\qquad h = \frac{2-1}{4} = \frac{1}{4}$

	x	$y = x^2$	
y_0	1	1^2	=1
y_1	$\frac{5}{4}$	$\left(\frac{5}{4}\right)^2$	=1.5625
y_2	$\frac{6}{4}$	$\left(\frac{6}{4}\right)^2$	=2.25
y_3	$\frac{7}{4}$	$\left(\frac{7}{4}\right)^2$	=3.0625
y_4	2	2^2	=4

$$I \approx \frac{1}{8}(1 + 2(1.5625 + 2.25 + 3.0625) + 4)$$

$$= \frac{1}{8}(18.75)$$

$$= 2.34375$$

2) Find the value of the integral $\int_0^{\frac{2\pi}{3}} \sqrt{\sin(x)}\,\mathrm{d}x$ using the trapezium rule with

(i) 4 intervals (ii) 8 intervals giving your answers to 3 dp

Note we must work to at least 4 dp if the answer is required to 3dp.

i) with 4 intervals

first we need to work out *h*

$$h = \frac{\frac{2\pi}{3} - 0}{4} \qquad = \frac{\pi}{6}$$

Now we need a table of values of $\sqrt{\sin(x)}$ for $x = 0$ to $\frac{2\pi}{3}$ in steps of $\frac{\pi}{6}$. These will be the values of y_0, y_1, y_2, y_3, and y_4.

x values		$\sqrt{\sin(x)}$	
0	y_0	$\sqrt{\sin(0)}$	0.0000
$\frac{\pi}{6}$	y_1	$\sqrt{\sin(\frac{\pi}{6})}$	0.7071
$\frac{2\pi}{6}$	y_2	$\sqrt{\sin(\frac{2\pi}{6})}$	0.9306
$\frac{3\pi}{6}$	y_3	$\sqrt{\sin(\frac{3\pi}{6})}$	1
$\frac{4\pi}{6} = \frac{2\pi}{3}$	y_4	$\sqrt{\sin(\frac{4\pi}{6})}$	0.9306

Then using formula for trapezium rule

$$\int_0^{\frac{2\pi}{3}} \sqrt{\sin(x)}\,\mathrm{d}x \approx \frac{h}{2}\left(y_0 + 2\left(y_1 + y_2 + y_3\right) + y_4\right)$$

gives $= \frac{\frac{\pi}{6}}{2}\left(0 + 2\left(0.7071 + 0.9306 + 1\right) + 0.9306\right) \qquad = \frac{\frac{\pi}{6}}{2} \times 6.2060$

=1.625 to 3dp

ii) with 8 intervals

first we need to work out *h*

$$h = \frac{\frac{2\pi}{3} - 0}{8} \qquad = \frac{\pi}{12}$$

Now we need a table of values of $\sqrt{\sin(x)}$ for $x = 0$ to $\frac{2\pi}{3}$ in steps of $\frac{\pi}{12}$. These will be the values of y_0, y_1, y_2, y_3, ... y_8 .

x values		$\sqrt{\sin(x)}$	
0	y_0	$\sqrt{\sin(0)}$	0.0000
$\frac{\pi}{12}$	y_1	$\sqrt{\sin(\frac{\pi}{12})}$	0.5087
$\frac{2\pi}{12}$	y_2	$\sqrt{\sin(\frac{\pi}{6})}$	0.7071
$\frac{3\pi}{12}$	y_3	$\sqrt{\sin(\frac{\pi}{4})}$	0.8409
$\frac{4\pi}{12}$	y_4	$\sqrt{\sin(\frac{\pi}{3})}$	0.9306
$\frac{5\pi}{12}$	y_5	$\sqrt{\sin(\frac{5\pi}{12})}$	0.9828
$\frac{6\pi}{12}$	y_6	$\sqrt{\sin(\frac{\pi}{2})}$	1
$\frac{7\pi}{12}$	y_7	$\sqrt{\sin(\frac{7\pi}{12})}$	0.9828
$\frac{8\pi}{12} = \frac{2\pi}{3}$	y_8	$\sqrt{\sin(\frac{8\pi}{12})}$	0.9306

Then using formula for trapezium rule

$$\int_0^{\frac{2\pi}{3}} \sqrt{\sin(x)}\,\mathrm{d}x \approx \frac{h}{2}\left(y_0 + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right) + y_8\right)$$

gives =

$$\frac{\frac{\pi}{12}}{2}\left(0 + 2\left(0.5087 + 0.7071 + 0.8409 + 0.9306 + 0.9828 + 1 + 0.9828\right) + 0.9306\right)$$

$$= \frac{\frac{\pi}{12}}{2} \times 12.8364$$

=1.680 to 3dp