

Lec.1/ Matrices, Inverse matrices by elementary row

- Determinant
- Determinant is a value that can be calculated from the elements of a **square matrix**. The determinant of a matrix A is denoted $\det(A)$, or the symbol for determinant is two vertical lines either side, $|A|$ means the determinant of A.
- It used to find the inverse of a matrix and useful in calculus for several applications.
- It used to check whether or not a matrix can be inverted, where if $\det(A)=0$ then there is no inverse.

- The calculation of determinant is as follows:

For 2x2 matrix,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex1: find the determinant of A

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 1(5) - 3(-2) = 5 + 6 = 11$$

For 3x3 matrix,

To each element of a 3x3 matrix there corresponds a 2x2 matrix that is obtained by deleting the row and column of that element. The determinate of the 2x2 matrix is called the minor of that element.

$$\begin{aligned} \bullet \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

- Ex2: find the determinant of A

$$\bullet \begin{vmatrix} 3 & 8 & 1 \\ 6 & 2 & -1 \\ -1 & -4 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} - 8 \begin{vmatrix} 6 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ -1 & -4 \end{vmatrix}$$

$$\bullet = 3(2-4) - 8(6-1) + (-24+2) = -68$$

- Notice the + - + pattern for the numbers of the first row.

- **Finding the inverse of matrices by elementary row method:** Also called the Gauss Jordan elimination method.
- Construct the augmented matrix ($A : I$)
- Using row operations: Change the rows using (1) adding or subtracting the row by another row, 2) multiplying the row by a constant and 3) swapping rows) until convert matrix A into the Identity Matrix I , ($I: A^{-1}$)

Note:

- 1) *Augmented matrices appear in Linear algebra as two appended matrices and are useful for solving systems of linear equations.*
- 2) *It can check the result through multiplying the original matrix by the inverse matrix to get the identity matrix ($A A^{-1} = I$)*

- Ex1: Find A^{-1} using elementary row method (Gaussian elimination)

$$\bullet A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \rightarrow R_1 = \frac{1}{2}R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & 1 \end{array} \right]$$

$$\bullet R'_2 = R_2 - R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{7}{2} & \frac{-1}{2} & 1 \end{array} \right], \quad R'_2 = \frac{2}{7}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array} \right]$$

$$\bullet R'_1 = R_1 - \frac{1}{2}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{7} & \frac{-1}{7} \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array} \right], \quad \rightarrow A^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

$$\bullet A A^{-1} = I \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

• Ex2: find A^{-1}

• $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$

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• $\begin{bmatrix} 2 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow R_1 = \frac{1}{2}R_1 \rightarrow \begin{bmatrix} \mathbf{1} & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$

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• $\rightarrow R_2 = R_2 - R_1 \rightarrow \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ \mathbf{0} & \frac{1}{2} & \frac{-7}{2} & \frac{-1}{2} & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$

• $\rightarrow R_3 = R_3 - 4R_1 \rightarrow \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-7}{2} & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{bmatrix}$

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• $\rightarrow R_2 = 2R_2 \rightarrow \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{bmatrix}$

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• $\rightarrow R_1 = \frac{1}{2}R_2 + R_1, R_3 = -2R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 0 & \frac{-4}{2} & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 10 & 0 & -4 & 1 \end{bmatrix}$

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• $\rightarrow R_3 = \frac{1}{10}R_3 \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$

• $\rightarrow R_1 = 2R_3 + R_1, R_2 = R_2 + 7R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & 0 & 1 & 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$

• $A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$

- **Homework:** find A^{-1} using Gaussian elimination and check the result

$$1) A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix} \quad \text{Ans: } A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 & \frac{1}{2} \\ 1 & -1 & 0 \end{bmatrix}$$

$$2) A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \quad \text{Ans: } A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$