

Newton-Raphson Method

It is based on linearization of the nonlinear continuous function $f(x)$. That is, the zero of $f(x)$ is approximated by the zero of the tangent line of $f(x)$.

Graphical Derivation of Newton-Raphson Method

Assume we have the nonlinear continuous function $y = f(x) = 0$ shown in Fig.1 and it is required to find its root (r). Let x_0 be the initial estimate of r

$$\tan \theta = \frac{dy}{dx} \Big|_{(x=x_0)} = f'(x = x_0) = \frac{f(x_0)}{x_0 - x_1}$$

Solve for x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Follow the same procedure to find x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

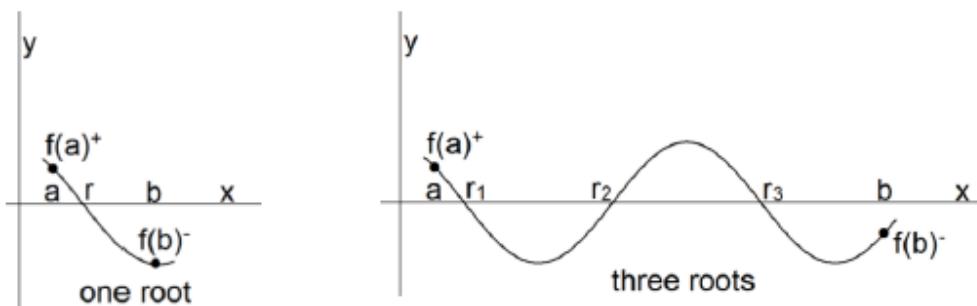
In general use the first approximation to get a second, the second to get a third, and so on, using the following numerical scheme

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad i = 0, 1, 2, \dots \quad f'(x_i) \neq 0 \quad (1)$$

The stopping criterion is: $|x_{i+1} - x_i| \leq T_x$ where T_x is the tolerance for x

- 1. Open Methods:** Sometimes we need to find the root of an equation near a point x . Use $x_0 = x$.
- 2. Bracketing methods:** Sometimes we need to find the root(s) of an equation in an interval $[a, b]$. Examine the sign of $f(x)$ at the ends of the interval $[a, b]$. There are two cases:

- If $f(a)f(b) < 0$, then there is one root or odd number of roots.



- If $f(a)f(b) > 0$, then there are no roots, even number of roots, or multiple equal roots.

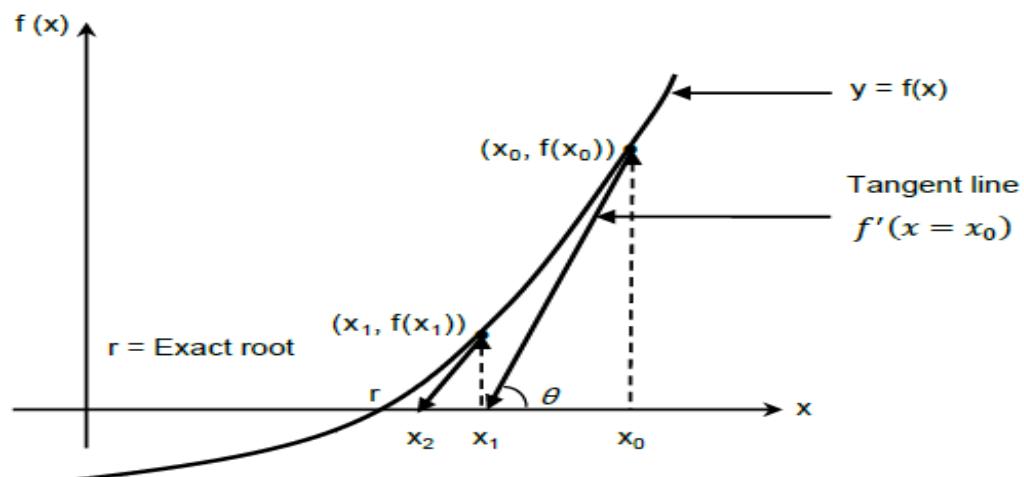
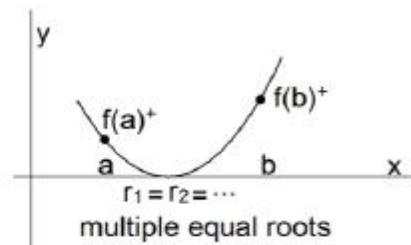
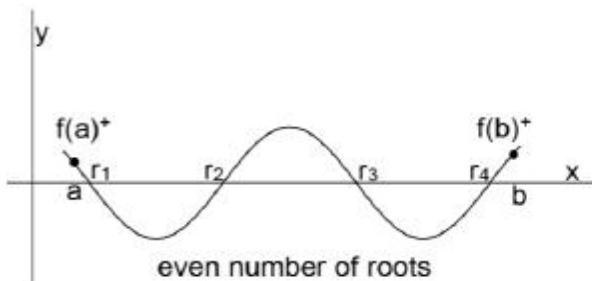
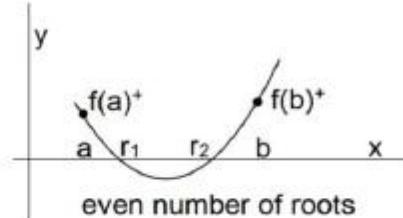
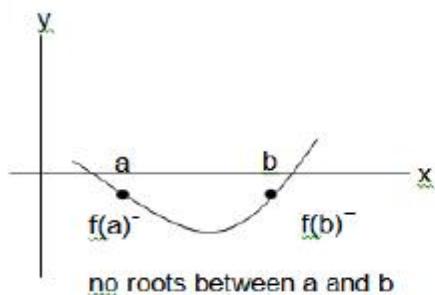
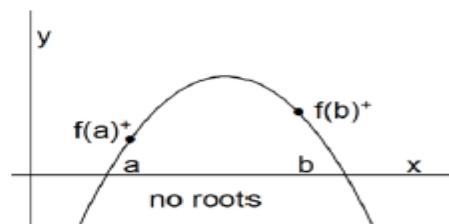
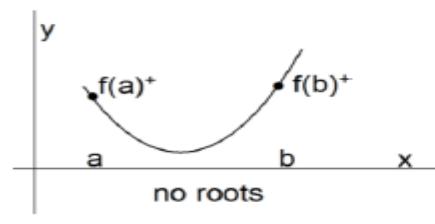


Figure 1: Geometrical illustration of Newton-Raphson method



Example 1: Use NR method to find the real root of $x^3 - x - 1 = 0$ correct to 5 decimal places (dp) in the interval $[-4, 4]$. Choose $\Delta x = 1$.

Solution

Determine the position of the root

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = x^3 - x - 1$	-	-	-	-	-	-	↓ r	+	+

Determine the initial point x_0 . The best initial point is the point that makes the value of $f(x)$ closer to zero.

x	1(x_1)	2(x_2)	1.5($x_3 = (x_1 + x_2)/2$)
$f(x) = x^3 - x - 1$	-1	5	0.875

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i)^3 - x_i - 1}{3(x_i)^2 - 1}$$

i	x_i (x old)	x_{i+1} (x new)	$E_i = x_{i+1} - x_i $	Notes
0	1.50000(x_0)	1.34783(x_1)	$E_0 = 0.15217 > T_x$	-
1	1.34783(x_1)	1.32520(x_2)	$E_1 = 0.02263 > T_x$	$E_1 < E_0$ ok
2	1.32520(x_2)	1.32472(x_3)	$E_2 = 0.00048 > T_x$	$E_2 < E_1$ ok
3	1.32472(x_3)	1.32472(x_4)	$E_3 = 0.00000 = T_x$	$E_3 < E_2$ ok

The root is $r = 1.32472$ and $f_{(x=1.32472)} = 0.00001$ to 5dp

Example 2: Use NR method to estimate the positive **abscissa** (x-coordinate) of the intersection point of $f_1(x) = \sin x$ and $f_2(x) = x^2$ correct to 4dp.

Solution

The intersection point (points) of $f_1(x)$ and $f_2(x)$ is the exact root (roots) of the equation $f(x) = f_1(x) - f_2(x) = 0$ and vice versa.

Determine the root position

x	0	0.5	1
$f(x) = \sin x - x^2$	0	+	$\downarrow r$

Determine the initial point x_0 . The best initial point is the point that makes the value of $f(x)$ closer to zero.

x	$0.5(x_1)$	$1(x_2)$	$0.75 (x_3 = (x_1 + x_2)/2)$
$f(x) = \sin x - x^2$	0.229	-0.159	0.119

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\sin x_i - x_i^2}{\cos x_i - 2x_i}$$

i	x_i (x old)	x_{i+1} (x new)	$E_i = x_{i+1} - x_i $	Notes
0	0.7500(x_0)	0.9051(x_1)	$E_0 = 0.1551$	-
1	0.9051(x_1)	0.8777(x_2)	$E_1 = 0.0274$	$E_1 < E_0$ OK
2	0.8777(x_2)	0.8767(x_3)	$E_2 = 0.0010$	$E_2 < E_1$ OK
3	0.8767(x_3)	0.8767 (x_4)	$E_2 = 0.0000$	$E_3 < E_2$ OK