Systems of equations

Consider the system of n linear equations and n unknowns, given by

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n.
\end{aligned}
$$

We can write this system as the matrix equation

$$\mathrm{A}\mathbf{x} = \mathbf{b},$$

with

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Gaussian Elimination

The standard numerical algorithm to solve a system of linear equations is called Gaussian Elimination. We can illustrate this algorithm by example.

Consider the system of equations

$$
\begin{aligned}
-3x_1 + 2x_2 - x_3 &= -1, \\
6x_1 - 6x_2 + 7x_3 &= -7, \\
3x_1 - 4x_2 + 4x_3 &= -6.
\end{aligned}
$$

To perform Gaussian elimination, we form an Augmented Matrix by combining the matrix A with the column vector $\mathbf{b}$:

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{pmatrix}.$$

Row reduction is then performed on this matrix. Allowed operations are (1) multiply any row by a constant, (2) add multiple of one row to another row, (3) interchange the order of any rows. The goal is to convert the original matrix into an upper-triangular matrix.

We start with the first row of the matrix and work our way down as follows. First we multiply the first row by 2 and add it to the second row, and add the first row to the third row:

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix}.$$

We then go to the second row. We multiply this row by -1 and add it to the third row:

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

The resulting equations can be determined from the matrix and are given by

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1 \\ -2x_2 + 5x_3 &= -9 \\ -2x_3 &= 2. \end{aligned}$$

These equations can be solved by backward substitution, starting from the last equation and working backwards. We have

$$\begin{aligned} &-2x_3 = 2 \rightarrow x_3 = -1 \\ &-2x_2 = -9 - 5x_3 = -4 \rightarrow x_2 = 2, \\ &-3x_1 = -1 - 2x_2 + x_3 = -6 \rightarrow x_1 = 2. \end{aligned}$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

LU decomposition

The process of Gaussian Elimination also results in the factoring of the matrix A to

$$A = LU,$$

where L is a lower triangular matrix and U is an upper triangular matrix. Using the same matrix A as in the last section, we show how this factorization is realized. We have

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_1A,$$

where

$$M_1A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix}.$$

Note that the matrix M_1 performs row elimination on the first column. Two times the first row is added to the second row and one times the first row is added to the third row. The entries of the column of M_1 come from $2 = -(6/-3)$ and $1 = -(3/-3)$ as required for row elimination. The number -3 is called the pivot.

The next step is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_2(M_1A),$$

where

$$M_2(M_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}.$$

Here, M_2 multiplies the second row by $-1 = -(-2/-2)$ and adds it to the third row. The pivot is -2.

We now have

$$M_2M_1A = U$$

or

$$A = M_1^{-1}M_2^{-1}U.$$

The inverse matrices are easy to find. The matrix M_1 multiples the first row by 2 and adds it to the second row, and multiplies the first row by 1 and adds it to the third row. To invert these operations, we need to multiply the first row by -2 and add it to the second row, and multiply the first row by -1 and add it to the third row. To check, with

$$M_1M_1^{-1} = I,$$

we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore,

$$L = M_1^{-1}M_2^{-1}$$

is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

With $\mathbf{y} = U\mathbf{x}$, we first solve $L\mathbf{y} = \mathbf{b}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

Using forward substitution

$$\begin{aligned} y_1 &= -1, \\ y_2 &= -7 + 2y_1 = -9, \\ y_3 &= -6 + y_1 - y_2 = 2. \end{aligned}$$

We now solve $U\mathbf{x} = \mathbf{y}$, that is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix}.$$

Using backward substitution,

$$
\begin{aligned}
&-2x_3 = 2 \to x_3 = -1, \\
&-2x_2 = -9 - 5x_3 = -4 \to x_2 = 2, \\
&-3x_1 = -1 - 2x_2 + x_3 = -6 \to x_1 = 2,
\end{aligned}
$$

and we have once again determined

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.
$$