The rank of a matrix

Linear dependence

To decide if a set of m-vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ are linearly independent, we have to solve the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

We know that $\mathbf{x} = \mathbf{0}$ is one solution, the *trivial solution*. Are there other (non-trivial) solutions?

- If yes, then the vectors are *linearly dependent*. We can use a non-trivial solution to express one vector as a linear combination of the others.
- If no, then the vectors are *linearly independent*

Linear systems and vector equations

A linear system of m equations is the same as a single vector equation of m-vectors. We may therefore re-write a vector equation as a linear system, and also re-write a linear system as a vector equation.

Example

Write the following linear system as a vector equation:

$$\begin{array}{ccccccc} 2x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 4x_1 & & & + & 2x_3 & = & 0 \\ & & 6x_2 & - & 3x_3 & = & 0 \end{array}$$

Linear systems and vector equations

Solution

We re-write the three equations as one equation of 3-vectors:

$$x_1 \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may write this as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$$

Note that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of the coefficient matrix of the linear system, and $\mathbf{0}$ is the last (augmented) column of the augmented matrix.

Criterion for linear independence

Theorem

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n-vectors, and let A be the $n \times n$ matrix with these vectors as columns. Then $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are linearly independent if and only if

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

Idea for proof: The linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution (that is, only the trivial solution) if and only if $\det(A) \neq 0$.

An example

Example

Show that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *are linearly independent when*

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Solution

Since we have

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{vmatrix} = (-4) \cdot 0 + (-2) \cdot 12 = -24 \neq 0$$

it follows that the vectors are linearly independent.

Rank of a matrix

Let A be any $m \times n$ matrix. Then A consists of n column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, which are m-vectors.

Definition

The rank of A is the maximal number of linearly independent column vectors in A, i.e. the maximal number of linearly independent vectors among $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. If $A = 0$, then the rank of A is 0.

We write $\mathrm{rk}(A)$ for the rank of A. Note that we may compute the rank of any matrix — square or not.

Rank of 2×2 matrices

Let us first see how to compute the rank of a 2×2 matrix:

Example

The rank of a 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is given by:

- $\text{rk}(A) = 2$ *if* $\det(A) = ad - bc \neq 0$*, since both column vectors are independent in this case*
- $\text{rk}(A) = 1$ *if* $\det(A) = 0$ *but* $A \neq 0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$*, since both column vectors are not linearly independent, but there is a single column vector that is linearly independent (i.e. non-zero)*
- $\text{rk}(A) = 0$ *if* $A = 0$

How do we compute $\text{rk}(A)$ for an $m \times n$ matrix A?

Computing rank using Gauss elimination

Gauss elimination

Use elementary row operations to reduce A to echelon form. The rank of A is the number of pivots or leading coefficients in the echelon form. In fact, the pivot columns (i.e. the columns with pivots in them) are linearly independent.

Note that it is not necessary to find the reduced echelon form — any echelon form will do since only the pivots matter.

Possible ranks

Counting possible number of pivots, we see that

- $\text{rk}(A) \leq m$ and $\text{rk}(A) \leq n$

for any $m \times n$ matrix A.

Rank: Example using Gauss elimination

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

We use elementary row operations:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Since the echelon form has pivots in the first three columns, A has rank $\mathrm{rk}(A) = 3$. *The first three columns of A are linearly independent.*