2) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

where $$\rho(x)=e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(x)$.

iii. Find the integrating factor $\rho(x)=e^{\int P(x)dx}$.

iv. Find y using the following equation

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example

Solve the equation $x\frac{dy}{dx}-3y=x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions P and Q***. To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx}-\frac{3}{x}y=x \qquad \Rightarrow \qquad P(x)=-\frac{3}{x}, \qquad Q(x)=x.$$

Step 2: ***Find an anti-derivative of*** $P(x)$.

$$\int P(x)dx = \int -\frac{3}{x}dx = -3\int \frac{1}{x}dx = -3\ln(x)$$

Step 3: ***Find the integrating factor*** $\rho(x)$.

$$\rho(x) = e^{\int P(x)dx} = e^{-3\ln x} = e^{\ln x^{-3}} = e^{\ln\frac{1}{x^3}} = \frac{1}{x^3}$$

Step 4: ***Find the solution***.

$$y = \frac{1}{\rho(x)}\int \rho(x)Q(x)dx = \frac{1}{(1/x^3)}\int\left(\frac{1}{x^3}\right)(x)dx$$

$$= x^3\int\frac{1}{x^2}dx = x^3\left(-\frac{1}{x}+C\right) = Cx^3 - x^2$$

The solution is the function $y = Cx^3 - x^2$.

Example

Solve the equation $(1+x^2)dy + (y - \tan^{-1}(x))dx = 0$.

Solution

Dividing the two sides by $(1+x^2)dx$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}(x)}{1+x^2} = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}(x)}{1+x^2} \quad \Rightarrow \quad P(x) = \frac{1}{1+x^2}, \quad Q = \frac{\tan^{-1}(x)}{1+x^2}$$

$$\int P(x)dx = \int\frac{1}{1+x^2}dx = \tan^{-1}(x)$$

$$\rho(x) = e^{\tan^{-1}(x)}$$

$$e^{\tan^{-1}(x)}y = \int e^{\tan^{-1}(x)}\frac{\tan^{-1}(x)}{1+x^2}dx + C$$

$$z = \tan^{-1}(x) \quad \Rightarrow \quad dz = \frac{1}{1+x^2}dx$$

$$e^{\tan^{-1}(x)}y = \int e^{z} \times z dz + C$$

$$= ze^{z} - \int e^{z} dz + C$$

$$= ze^{z} - e^{z} + C$$

$$e^{\tan^{-1}(x)}y = \tan^{-1}(x)e^{\tan^{-1}(x)} - e^{\tan^{-1}(x)} + C$$

Steps for Solving other Form of Linear First Order Equation

There is another form of differential equation that can be written in the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where P and Q are functions of y. The solution is found as follows:

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(y)$.

iii. Find the integrating factor $\rho(y) = e^{\int P(y)dy}$.

iv. Find x using the following equation

$$x = \frac{1}{\rho(y)} \int \rho(y)Q(y)dy$$

Example

Solve the equation $e^{2y}dx + 2(xe^{2y} - y)dy = 0$.

Solution

Dividing the differential equation by $e^{2y}dy$ to get

$$\frac{dx}{dy} + 2x - 2ye^{-2y} = 0$$

$$\frac{dx}{dy} + 2x = 2ye^{-2y} \quad \Rightarrow \quad P(y) = 2, \quad Q(y) = 2ye^{-2y}$$

$$\int P(y)dy = \int 2dy = 2y, \quad \rho(y) = e^{\int P(y)dy} = e^{2y}$$

$$x = \frac{1}{e^{2y}} \int (e^{2y})(2ye^{-2y})dy + C \quad \Rightarrow \quad e^{2y}x = 2\int ydy + C$$

$$e^{2y}x = 2\frac{y^2}{2} + C \quad \Rightarrow \quad e^{2y}x = y^2 + C$$

Reduciblе to Linear

- The general form

$$\frac{dy}{dx} + P(x)y = Q(x)f(y)$$

where the function f is y to any power n.

- Also, it may be in the following form

$$\frac{dy}{dx} + P(x)g(y) = Q(x)h(y)$$

where the function g and h are functions of y.

Example

Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \ln(x)y^2$

Solution

Dividing the two sides of the equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \ln(x)$$

Let $z = \frac{1}{y} \quad \Rightarrow \quad \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -y^2\frac{dz}{dx}$

$$-\frac{dz}{dx} + \frac{1}{x}z = \ln(x)$$

$$\frac{dz}{dx} - \frac{1}{x}z = -\ln(x) \qquad \Rightarrow \quad P = \frac{-1}{x}, \quad Q = -\ln(x)$$

$$\int P(x)dx = \int \frac{-1}{x}dx = -\ln(x)$$

$$\rho(x) = e^{\int P(x)dx} = e^{-\ln(x)} = e^{\ln(x)^{-1}} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$\frac{1}{x}z = -\int \frac{1}{x}\ln(x)dx + C$$

$$\frac{1}{x}\times\frac{1}{y} = -\frac{(\ln(x))^2}{2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{(\ln(x))^2}{2} + C$$

Example

Solve the equation $\frac{dy}{dx} + x\sin(2y) = x\cos^2(y)$

Solution

Dividing the two sides of the equation by $\cos^2(y)$

$$\frac{1}{\cos^2(y)}\frac{dy}{dx} + x\frac{\sin(2y)}{\cos^2(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)\cos(y)}{\cos^2(y)} = x$$

$$\sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)}{\cos(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + 2x\tan(y) = x$$

Let $z = \tan(y) \quad \Rightarrow \quad \frac{dz}{dx} = \sec^2(y)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}\frac{dz}{dx}$

$$\frac{dz}{dx} + 2xz = x \quad \Rightarrow \quad P = 2x, \quad Q = x$$

$$\int P(x)dx = \int 2xdx = x^2 \quad \Rightarrow \quad \rho(x) = e^{\int P(x)dx} = e^{x^2}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$e^{x^2}z = \int e^{x^2}(x)dx + C \quad \Rightarrow \quad e^{x^2}\tan(y) = \frac{e^{x^2}}{2} + C$$

Another Form of Reducible to Linear

- The general form may be as follows

$$\frac{dx}{dy}+P(y)x=Q(y)f(x)$$

where the function f is x to any power n.

- Also, it may be in the following form

$$\frac{dx}{dy}+P(y)g(x)=Q(y)h(x)$$

where the function g and h are functions of x.

Example

Solve the equation $\cos(y)dx=x(\sin(y)-x)dy$

Solution

Dividing the two sides of the equation by $\cos(y)dy$

$$\frac{dx}{dy}=\frac{\sin(y)}{\cos(y)}x-\frac{x^2}{\cos(y)} \quad \Rightarrow \quad \frac{dx}{dy}-x\tan(y)=-x^2\sec(y)$$

Dividing by x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}\tan(y)=-\sec(y)$$

Let $z=\frac{1}{x} \quad \Rightarrow \quad \frac{dz}{dy}=\frac{-1}{x^2}\frac{dx}{dy} \quad \Rightarrow \quad \frac{dx}{dy}=-x^2\frac{dz}{dy}$

$$-\frac{dz}{dy}-z\tan(y)=-\sec(y)$$

$$\frac{dz}{dy}+z\tan(y)=\sec(y) \quad \Rightarrow \quad P=\tan(y), \quad Q=\sec(y)$$

$$\int P(y)dy = \int \tan(y)dy = \int \frac{\sin(y)}{\cos(y)}dy = -\ln(\cos(y))$$

$$\rho(y) = e^{\int P(y)dy} = e^{-\ln(\cos(y))} = e^{\ln(\cos(y))^{-1}} = e^{\ln\left(\frac{1}{\cos(y)}\right)} = \sec(y)$$

$$\rho(y)z = \int \rho(y)Q(y)dy + C$$

$$\sec(y) \times \frac{1}{x} = \int \sec(y)\sec(y)dy + C$$

$$\frac{\sec(y)}{x} = \int \sec^2(y)dy + C \quad \Rightarrow \quad \frac{\sec(y)}{x} = \tan(y) + C$$

Exact Differential Equations

Example

If $f(x,y)=C$ and $f(x,y)=\sin(xy)$ then

$$\frac{df}{dx}=y\cos(xy)+x\cos(xy)\frac{dy}{dx}=0, \text{ or}$$

$$df=y\cos(xy)dx+x\cos(xy)dy=0$$

i.e., $y\cos(xy)dx+x\cos(xy)dy=0$

From the above equation, we see that $M(x,y)=y\cos(xy)=\dfrac{\partial f}{\partial x}$, and $N(x,y)=x\cos(xy)=\dfrac{\partial f}{\partial y}$. The solution of this differential equation is $f(x,y)=C$.

Exact Differential Equation Test

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be ***exact*** if for some function $f(x,y)$

$$M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=df$$

is exact if and only if $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Example

- The equation $(x^2+y^2)dx+(2xy+\cos(y))dy=0$ is exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)=2y, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xy+\cos(y))=2y$$

are equal.

- The equation $(x+3y)dx+(x^2+\cos(y))dy=0$ is not exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x+3y)=3, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2+\cos(y))=2x$$

are not equal.

Steps for Solving an Exact Differential Equation

i. Match the equation to the form $M(x,y)dx+N(x,y)dy=0$ to identify M and N.

ii. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$).

iii. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $g'(y)$ (or $g'(x)$).

iv. Integrate to find $g(y)$ (or $g(x)$).

v. Write the solution of the exact equation as $f(x,y)=C$.

Example

Solve the differential equation

$$(x^2+y^2)dx+(2xy+\cos(y))dy=0.$$

Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** M.

$$M(x,y)=x^2+y^2$$

Step 2: ***Integrate*** M ***with respect to*** x, ***writing the constant of integration as*** $g(y)$.

$$f(x, y) = \int M(x, y)dx = \int (x^2 + y^2)dx = \frac{x^3}{3} + xy^2 + g(y)$$

Step 3: ***Differentiate with respect to*** y ***and set the result equal to*** N ***to find*** $g'(y)$.

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy^2 + g(y)\right) = 2xy + g'(y)$$

$$2xy + g'(y) = 2xy + \cos(y) \quad \Rightarrow \quad g'(y) = \cos(y)$$

Step 4: ***Integrate to find*** $g(y)$.

$$\int g'(y)dy = \int \cos(y)dy = \sin(y)$$

Step 5: ***Write the solution of the exact equation as*** $f(x, y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Another Solution

Step 1: ***Match the equation to the form*** $M(x, y)dx + N(x, y)dy = 0$ ***to identify*** N.

$$N(x, y) = 2xy + \cos(y)$$

Step 2: ***Integrate*** N ***with respect to*** y, ***writing the constant of integration as*** $g(x)$.

$$f(x, y) = \int N(x, y)dy = \int (2xy + \cos(y))dy = xy^2 + \sin(y) + g(x)$$

Step 3: ***Differentiate with respect to*** x ***and set the result equal to*** M ***to find*** $g'(x)$.

$$\frac{\partial}{\partial x}\left(xy^2 + \sin(y) + g(x)\right) = y^2 + g'(x)$$

$$y^2 + g'(x) = x^2 + y^2 \quad \Rightarrow \quad g'(x) = x^2$$

Step 4: ***Integrate to find*** $g(x)$.

$$\int g'(x)dx = \int x^2 dx = \frac{x^3}{3}$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Reducible to Exact

A differential equation $M(x,y)dx + N(x,y)dy = 0$ which is not exact can be made exact by multiplying both sides by a suitable integrating factor ρ. In other words, the equation

$$\rho M(x,y)dx + \rho N(x,y)dy = 0$$

is an exact equation for an appropriate choice of ρ.

Method to Find the Integrating Factor

- If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ or ***Constant*** then $\rho(x) = e^{\int f(x)dx}$.
- If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ or ***Constant*** then $\rho(y) = e^{\int f(y)dy}$.

Example

Solve the equation $2ydx + xdy = 0$

Solution

$$M(x,y) = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N(x,y) = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

This equation is not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x} = f(x)$$

$$\int f(x)dx = \int \frac{1}{x}dx = \ln(x)$$

$$\rho(x) = e^{\int f(x)dx} = e^{\ln(x)} = x$$

Multiplying both sides of the equation by the integrating factor $\rho(x) = x$, we get

$$x(2ydx + xdy) = 0 \quad \Rightarrow \quad 2xydx + x^2dy = 0$$

which is exact because $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and the solution is

$$f(x,y) = \int 2xydx = x^2y + g(y)$$

$$\frac{\partial}{\partial y}\left(x^2y + g(y)\right) = x^2 + g'(y)$$

$$x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = \int g'(y)dy = C \quad \Rightarrow \quad x^2y = C$$