Inverse of Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$, Then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{K}{s}$	K
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$ n: positive & integer
$\frac{1}{s \pm a}$	$e^{\mp at}$
$\frac{1}{s^2+a^2}$	$\frac{1}{a} \sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a} \sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{s^2}$	t
$F(s \mp a)$	$e^{\pm at} f(t)$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$

$\mathcal{L}^{-1}\{\ \}$	$f(t)$
$\mathcal{L}^{-1}\left\{ \dfrac{dF(s)}{ds} \right\}$	$-t\,f(t)$
$\mathcal{L}^{-1}\left\{ \dfrac{F(s)}{s} \right\}$	$\int_0^t f(t)\,dt$
$\mathcal{L}^{-1}\left\{ \dfrac{1}{s} e^{-as} \right\}$	$u(t-a)$
$\mathcal{L}^{-1}\left\{ F(s)\, e^{-as} \right\}$	$f(t-a)\,u(t-a)$

Ex: ① $\mathcal{L}^{-1}\left\{\dfrac{2s-18}{s^2+9}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\dfrac{18}{s^2+9}\right\}$

$= 2\cos 3t - 6\sin 3t$

Ex ② $\mathcal{L}^{-1}\left\{\dfrac{(s+1)^2}{(s+2)^4}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{[(s+2)-2+1]^2}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{[(s+2)-1]^2}{(s+2)^4}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{(s+2)^2-2(s+2)+1}{(s+2)^4}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2} - \dfrac{2}{(s+2)^3} + \dfrac{1}{(s+2)^4}\right\}$

$e^{-2t}\left[t - \dfrac{2}{2!}t^2 + \dfrac{1}{3!}t^3\right]$

Ex③ $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s+5}\right\}$

Soln

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2-1+5}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$=e^{t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=e^{t}\,\frac{1}{2}\sin 2t$$

لازم معامل x^2 واحد

Note

$\boxed{\text{II}}\ x^2+ax+b$

$\rightarrow \left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+b$ اكمال المربع

Ex $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-2s+5}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-1)^2+4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{(s-1)+1}{(s-1)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= e^{t}\cos 2t + \frac{1}{2}e^{t}\sin 2t$$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^5}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{(s+1)-1}{(s+1)^5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^4} - \frac{1}{s^5}\right\}$$

$$= e^{-t}\left[\frac{t^3}{3!} - \frac{t^4}{4!}\right]$$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s+18}{s^2+9} + \dfrac{24-30s}{s^4}\right\}$

Soln

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} - \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30}{s^3}\right\}$$

$$= 5t + 4\frac{t^2}{2} - 2\cos 3t - 6\sin 3t + \frac{24}{3!}t^3 - \frac{30}{2!}t^2$$

Ex: $\mathcal{L}^{-1}\left\{ \ln\left(1+\frac{4}{s^2}\right)\right\}$

Soln $f(t) = \mathcal{L}^{-1}\left\{ \ln\left(\frac{s^2+4}{s^2}\right)\right\}$

$= \mathcal{L}^{-1}\left\{ \ln(s^2+4) - \ln(s^2) \right\}$

$\mathcal{L}^{-1}\left\{ \frac{dF(s)}{ds} \right\} = -t\, f(t)$

$F(s) = \ln(s^2+4) - \ln s^2$

$\frac{dF(s)}{ds} = \frac{2s}{s^2+4} - \frac{2}{s}$

$\mathcal{L}^{-1}\left\{ \frac{dF(s)}{ds} \right\} = -t\, f(t)$

$= \left[2\cos 2t - 2 \right]$

$\therefore f(t) = -2\,\frac{(\cos 2t - 1)}{t}$

$f(t) = 2\cdot\frac{(1-\cos 2t)}{t}$

Ex: Find $\mathcal{L}^{-1}\{\cot^{-1}(s+1)\}$

Sol: $\mathcal{L}^{-1}\{\cot^{-1}(s+1)\} = e^{-t}\mathcal{L}^{-1}\{\cot^{-1} s\}$

* for $\mathcal{L}^{-1}\{\cot^{-1} s\}$

$$F(s) = \cot^{-1} s$$

$$\frac{dF(s)}{ds} = -\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = -t\,f(t)$$

$$= \sin t = t\,f(t)$$

$$\therefore f(t) = \frac{\sin t}{t}$$

$$\therefore \mathcal{L}^{-1}\{\cot^{-1}(s+1)\} = e^{-t}\,\frac{\sin t}{t}$$

Ex: Find $\mathcal{L}^{-1}\left\{\tan^{-1}\frac{3}{s}\right\}$

Soln $F(s) = \tan^{-1}\frac{3}{s}$

$$\frac{dF(s)}{ds} = \frac{1}{\left(\frac{3}{s}\right)^2 + 1}\left(\frac{-3}{s^2}\right)$$

$$\frac{dF(s)}{ds} = \frac{-3}{9 + s^2}$$

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\} = \mathcal{L}^{-1}\left\{\frac{-3}{9+s^2}\right\} = -t\,f(t)$$

$$= \cancel{-}\cancel{3}\,\frac{\sin 3t}{\cancel{3}} = \cancel{-}\cancel{t}\,f(t)$$

$$f(t) = \frac{\sin 3t}{t}$$

$$\therefore \mathcal{L}^{-1}\left\{\tan^{-1}\frac{3}{s}\right\} = \frac{\sin 3t}{t}$$

Ex: If $f(t) = \mathcal{L}^{-1}\left\{ \frac{3}{s} - \frac{4e^{-s}}{s^2} + 4\frac{e^{-3s}}{s^2} \right\}$

Find and sketch $f(t)$, evaluate $f(2)$, $f(7)$

Soln $f(t) = \mathcal{L}^{-1}\left\{ \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4}{s^2}e^{-3s} \right\}$

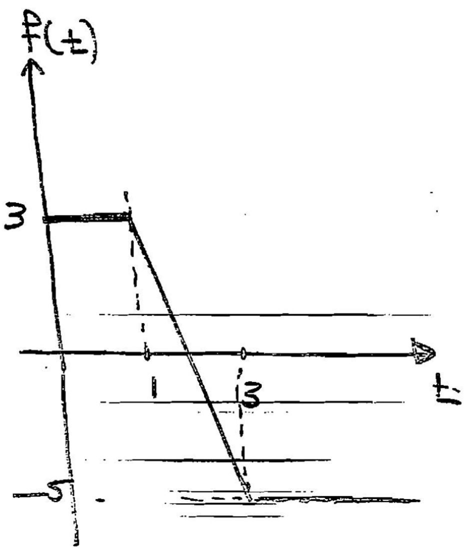
$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} = t$$

$$\mathcal{L}^{-1}\left\{ \frac{3}{s} - \frac{4}{s^2}e^{-s} + \frac{4}{s^2}e^{-3s} \right\}$$

$$= 3 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$$

$$f(t) = \begin{cases} 3 & 0 < t < 1 \\ 3-4(t-1) & 1 < t < 3 \\ 3-4(t-1)+4(t-3) & t > 3 \end{cases}$$

$$= \begin{cases} 3 & 0 < t < 1 \\ 7-4t & 1 < t < 3 \\ -5 & t > 3 \end{cases}$$



$\therefore f(2) = -1 \rightsquigarrow$ From 2nd Term

$\therefore f(7) = -5 \rightsquigarrow$ From 3rd Term

$$\therefore f(t) = \begin{cases} \frac{1}{2}t^2 & 0 < t < 2 \\ \frac{1}{2}t^2 - 2(t-2)^2 & 2 < t < 4 \\ \frac{1}{2}t^2 - 2(t-2)^2 + \frac{3}{2}(t-4)^2 & t > 4 \end{cases}$$

$f(1) = \frac{1}{2}(1)^2 = 1/2 \rightsquigarrow$ From 1st Term

$f(3) = \frac{1}{2}(3)^2 - 2(1)^2 = \frac{9}{2} - 2 = 2.5 \rightsquigarrow$ From 2nd Term

$f(5) = \frac{1}{2}(5)^2 - 2(3)^2 + \frac{3}{2}(1)^2 = \checkmark \rightsquigarrow$ From 3rd Term

Ex: $\mathcal{L}^{-1}\left\{\dfrac{e^{4-3s}}{(s+4)^4}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{e^{4}\,e^{-3s}}{(s+4)^4}\right\}$ $\rightarrow u(t-3)$

$$\mathcal{L}^{-1}\left\{\frac{e^4}{(s+4)^4}\right\} = e^4 e^{-4t}\frac{t^3}{3!} = \frac{1}{3!}e^{-4(t-1)}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^4}\right\} = e^{-4(t-1-3)}\frac{1}{3!}(t-3)^3\,u(t-3)$$

$$= e^{-4(t-4)}\frac{1}{3!}(t-3)^3\,u(t-3)$$

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{(s+2)e^{-\pi s}}{s^2+2s+2}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{(s+2)\,e^{-\pi s}}{s^2+2s+2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2-1+2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t + e^{-t}\sin t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{(s+2)e^{-\pi s}}{s^2+2s+2}\right\} = \left[e^{-(t-\pi)}(\cos(t-\pi)+\sin(t-\pi)\right]u(t-\pi)$$