

Chapter Three

Gamma & Beta Functions

I. Gamma Function

Definition

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx ; n > 0$$

$$\& \Gamma(n) = \Gamma(n+1) / n ; n \in \mathbb{R} - \mathbb{Z}^{\leq 0}$$

Results:

$$(1) \Gamma(n+1) = n \Gamma(n) ; n > 0 , \text{ where } \Gamma(1) = 1$$

$$(2) \Gamma(n+1) = n! ; n \in \mathbb{N} \quad (\text{convention: } 0! = 1)$$

$$(3) \Gamma(n) \Gamma(1-n) = \pi / \sin(n\pi) ; 0 < n < 1$$

In Particular;

$$\Gamma(1/2) = \sqrt{\pi}$$

Examples:

Example(1)

Evaluate $\int_0^{\infty} x^4 e^{-x} dx$

Solution

$$\int_0^{\infty} x^4 e^{-x} dx = \int_0^{\infty} x^{5-1} e^{-x} dx = \Gamma(5)$$

$$\Gamma(5) = \Gamma(4+1) = 4! = 4(3)(2)(1) = 24$$

Example(2)

Evaluate $\int_0^{\infty} x^{1/2} e^{-x} dx$

$$\int_0^{\infty} x^{1/2} e^{-x} dx = \int_0^{\infty} x^{3/2-1} e^{-x} dx = \Gamma(3/2)$$

$$3/2 = 1/2 + 1$$

$$\Gamma(3/2) = \Gamma(1/2 + 1) = 1/2 \Gamma(1/2) = 1/2 \sqrt{\pi}$$

Example(3)

Evaluate $\int_0^{\infty} x^{3/2} e^{-x} dx$

$$\int_0^{\infty} x^{3/2} e^{-x} dx = \int_0^{\infty} x^{5/2-1} e^{-x} dx = \Gamma(5/2)$$

$$5/2 = 3/2 + 1$$

$$\Gamma(5/2) = \Gamma(3/2 + 1) = 3/2 \Gamma(3/2) = 3/2 \cdot 1/2 \Gamma(1/2) = 3/2 \cdot 1/2 \cdot \sqrt{\pi} = 3/4 \sqrt{\pi}$$

Example(4)

Find $\Gamma(-1/2)$

$$(-1/2) + 1 = 1/2$$

$$\Gamma(-1/2) = \Gamma(-1/2 + 1) / (-1/2) = -2 \Gamma(1/2) = -2 \sqrt{\pi}$$

Example(5)

Find $\Gamma(-3/2)$

$$(-3/2) + 1 = -1/2$$

$$\Gamma(-3/2) = \Gamma(-3/2 + 1) / (-3/2) = \Gamma(-1/2) / (-2/3) = (-2 \sqrt{\pi}) / (-2/3) = 4 \sqrt{\pi} / 3$$

II. Beta Function

Definition

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx ; m > 0 \ \& \ n > 0$$

Results:

$$(1) \ B(m,n) = \Gamma(m) \Gamma(n) / \Gamma(m+n)$$

$$(2) \ B(m,n) = B(n,m)$$

$$(3) \ \int_0^{\pi/2} \sin^{2m-1}x \cdot \cos^{2n-1}x \ dx = \Gamma(m) \Gamma(n) / 2 \Gamma(m+n) ; m > 0 \ \& \ n > 0$$

$$(4) \ \int_0^{\infty} x^{q-1} / (1+x) \cdot dx = \Gamma(q) \Gamma(1-q) = \pi / \sin(q\pi) ; 0 < q < 1$$

Examples:

Example(1)

$$\text{Evaluate } \int_0^1 x^4 (1-x)^3 dx$$

Solution

$$\int_0^1 x^4 (1-x)^3 dx = \int_0^1 x^{5-1} (1-x)^{4-1} dx$$

$$= B(5,4) = \Gamma(5) \Gamma(4) / \Gamma(9) = 4! \cdot 3! / 8! = 3! / (8 \cdot 7 \cdot 6 \cdot 5) = 1 / (8 \cdot 7 \cdot 5) = 1/280$$

Example(2)

$$\text{Evaluate } I = \int_0^1 \left[1 / \sqrt[3]{x^2(1-x)} \right] dx$$

Solution

$$I = \int_0^1 x^{-2/3} (1-x)^{-1/3} dx = \int_0^1 x^{1/3-1} (1-x)^{2/3-1} dx$$

$$= B(1/3, 2/3) = \Gamma(1/3) \Gamma(2/3) / \Gamma(1)$$

$$\Gamma(1/3) \Gamma(2/3) = \Gamma(1/3) \Gamma(1 - 1/3) = \pi / \sin(\pi/3) = \pi / (\sqrt{3}/2) = 2\pi / \sqrt{3}$$

Example(3)

Evaluate $I = \int_0^1 \sqrt{x} \cdot (1-x) dx$

Solution

$$I = \int_0^1 x^{1/2} (1-x) dx = \int_0^1 x^{3/2-1} (1-x)^{2-1} dx \\ = B(3/2, 2) = \Gamma(3/2) \Gamma(2) / \Gamma(7/2)$$

$$\Gamma(3/2) = \frac{1}{2} \sqrt{\pi}$$

$$\Gamma(5/2) = \Gamma(3/2+1) = (3/2) \Gamma(3/2) = (3/2) \cdot \frac{1}{2} \sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(7/2) = \Gamma(5/2+1) = (5/2) \Gamma(5/2) = (5/2) \cdot (\frac{3\sqrt{\pi}}{4}) = \frac{15\sqrt{\pi}}{8}$$

Thus,

$$I = (\frac{1}{2} \sqrt{\pi}) \cdot 1! / (\frac{15\sqrt{\pi}}{8}) = 4/15$$

II. Using Gamma Function to Evaluate Integrals

Example(1)

Evaluate: $I = \int_0^{\infty} x^6 e^{-2x} dx$

Solution:

Letting $y = 2x$, we get

$$I = (1/128) \int_0^{\infty} y^6 e^{-y} dy = (1/128) \Gamma(7) = (1/128) 6! = 45/8$$

Example(2)

Evaluate: $I = \int_0^{\infty} \sqrt{x} e^{-x^3} dx$

Solution:

Letting $y = x^3$, we get

$$I = (1/3) \int_0^{\infty} y^{-1/2} e^{-y} dy = (1/3) \Gamma(1/2) = \sqrt{\pi} / 3$$

Example(3)

Evaluate: $I = \int_0^{\infty} x^m e^{-kx^n} dx$

Solution:

Letting $y = kx^n$, we get

$$I = [1 / (n \cdot k^{(m+1)/n})] \int_0^{\infty} y^{[(m+1)/n - 1]} e^{-y} dy = [1 / (n \cdot k^{(m+1)/n})] \Gamma[(m+1)/n]$$

II. Using Beta Function to Evaluate Integrals

Formulas

$$(1) \int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m,n) = \Gamma(m) \Gamma(n) / \Gamma(m+n) \quad ; m > 0 \text{ \& } n > 0$$

$$(3) \int_0^{\pi/2} \sin^{2m-1} x \cdot \cos^{2n-1} x dx = (1/2) B(m,n) \quad ; m > 0 \text{ \& } n > 0$$

$$(4) \int_0^{\infty} x^{q-1} / (1+x) \cdot dx = \Gamma(q) \Gamma(1-q) = \Pi / \sin(q\pi) \quad ; 0 < q < 1$$

Using Formula (1)

Example(1)

Evaluate: $I = \int_0^2 x^2 / \sqrt{2-x} \cdot dx$

Solution:

Letting $x = 2y$, we get

$$I = (8/\sqrt{2}) \int_0^1 y^2 (1-y)^{-1/2} dy = (8/\sqrt{2}) \cdot B(3, 1/2) = 64\sqrt{2} / 15$$

Example(2)

Evaluate: $I = \int_0^a x^4 \sqrt{a^2 - x^2} \cdot dx$

Solution:

Letting $x^2 = a^2 y$, we get

$$I = (a^6 / 2) \int_0^1 y^{3/2} (1 - y)^{1/2} dy = (a^6 / 2) \cdot B(5/2, 3/2) = a^6 / 32$$

Example(3)

Evaluate: $I = \int_0^2 x \sqrt{8 - x^3} \cdot dx$

Hint

Let $x^3 = 8y$

Answer

$$I = (8/3) \int_0^1 y^{-1/3} (1 - y)^{1/3} \cdot dy = (8/3) B(2/3, 4/3) = 16\pi / (9\sqrt{3})$$

Using Formula (3)

Example(3)

Evaluate: $I = \int_0^\infty dx / (1+x^4)$

Solution:

Letting $x^4 = y$, we get

$$\begin{aligned} I &= (1/4) \int_0^\infty y^{-3/4} dy / (1+y) = (1/4) \cdot \Gamma(1/4) \cdot \Gamma(1 - 1/4) \\ &= (1/4) \cdot [\pi / \sin(\frac{1}{4} \cdot \pi)] = \pi\sqrt{2} / 4 \end{aligned}$$

Using Formula (2)

Example(4)

a. Evaluate: $I = \int_0^{\pi/2} \sin^3 x \cdot \cos^2 x \, dx$

b. Evaluate: $I = \int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \, dx$

Solution:

a. Notice that: $2m - 1 = 3 \rightarrow m = 2$ & $2n - 1 = 2 \rightarrow m = 3/2$

$$I = (1/2) B(2, 3/2) = 8/15$$

b. $I = (1/2) B(5/2, 3) = 8/315$

Example(5)

a. Evaluate: $I = \int_0^{\pi/2} \sin^6 x \, dx$

b. Evaluate: $I = \int_0^{\pi/2} \cos^6 x \, dx$

Solution:

a. Notice that: $2m - 1 = 6 \rightarrow m = 7/2$ & $2n - 1 = 0 \rightarrow m = 1/2$

$$I = (1/2) B(7/2, 1/2) = 5\pi/32$$

b. $I = (1/2) B(1/2, 7/2) = 5\pi/32$

Example(6)

a. Evaluate: $I = \int_0^{\pi} \cos^4 x \, dx$

b. Evaluate: $I = \int_0^{2\pi} \sin^8 x \, dx$

Solution:

a. $I = \int_0^{\pi} \cos^4 x \, dx = 2 \int_0^{\pi/2} \cos^4 x \, dx = 2 (1/2) B(1/2, 5/2) = 3\pi / 8$

b. $I = \int_0^{2\pi} \sin^8 x \, dx = 4 \int_0^{\pi/2} \sin^8 x \, dx = 4 (1/2) B(9/2, 1/2) = 35\pi / 64$

Details

I.

Example(1)

Evaluate: $I = \int_0^{\infty} x^6 e^{-2x} \, dx$

$$x = y/2$$

$$x^6 = y^6 / 64$$

$$dx = (1/2)dy$$

$$x^6 e^{-2x} dx = y^6 / 64 e^{-y} \cdot (1/2)dy$$

Example(2)

$I = \int_0^{\infty} \sqrt{x} e^{-x^3} \, dx \quad x=y^{1/3}$

$$\sqrt{x} = y^{1/6}$$

$$dx = (1/3)y^{-2/3} dy$$

$$\sqrt{x} e^{-x^3} dx = y^{1/6} e^{-y} \cdot (1/3)y^{-2/3} dy$$

Example(3)

Evaluate: $I = \int_0^{\infty} x^m e^{-kx^n} \, dx$

$$y = kx^n$$

$$x = y^{1/n} / k^{1/n}$$

$$x^m = y^{m/n} / k^{m/n}$$

$$dx = (1/n) y^{(1/n-1)} / k^{1/n} dy$$

$$x^m e^{-kx^n} dx = (y^{m/n} / k^{m/n}) \cdot e^{-y} \cdot (1/n) y^{(1/n-1)} / k^{1/n} dy$$

$$m/n + 1/n - 1 = (m+1)/n - 1$$

$$-m/n - 1/n = -(m+1)/n$$

$$I = [1 / (n \cdot k^{(m+1)/n})] \int_0^{\infty} y^{[(m+1)/n - 1]} e^{-y} dy$$

II. Examples

Example(1)

$$I = \int_0^2 \frac{x^2}{\sqrt{2-x}} \cdot dx$$
$$x = 2y$$
$$dx = 2dy$$
$$x^2 = 4y^2$$

$$\sqrt{2-x} = \sqrt{2-2y} = \sqrt{2} \sqrt{1-y}$$

$$x^2 / \sqrt{2-x} \cdot dx = 4y^2 / \sqrt{2} \sqrt{1-y} \cdot 2dy$$

$$y=0 \text{ when } x=0$$

$$y=1 \text{ when } x=2$$

Example(2)

$$\text{Evaluate: } I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}} \cdot dx$$

$$x^2 = a^2 y, \text{ we get}$$

$$x^4 = a^4 y^2$$

$$x = a y^{1/2}$$

$$dx = (1/2)a y^{-1/2} dy$$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2 y} = a(1-y)^{1/2}$$

$$x^4 / \sqrt{a^2-x^2} \cdot dx = \frac{a^4 y^2}{a(1-y)^{1/2}} \cdot (1/2)a y^{-1/2} dy$$

$$y=0 \text{ when } x=0$$

$$y=1 \text{ when } x=a$$

Example(3)

$$I = \int_0^\infty \frac{dx}{(1+x^4)}$$

$$x^4 = y$$

$$x = y^{1/4}$$

$$dx = (1/4) y^{-3/4} dy$$

$$\frac{dx}{(1+x^4)} = \frac{(1/4) y^{-3/4} dy}{(1+y)}$$

Proofs of formulas (2) & (3)

Formula (2)

We have,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Let $x = \sin^2 y$

Then $dx = 2 \sin y \cos y dy$

&

$$\begin{aligned} x^{m-1} (1-x)^{n-1} dx &= (\sin^2 y)^{m-1} (\cos^2 y)^{n-1} (dy / 2 \sin y \cos y) \\ &= 2 \sin^{2m-1} y \cdot \cos^{2n-1} y dy \end{aligned}$$

When $x=0$, we have $y = 0$

When $x=1$, we have $y = \pi/2$

Thus,

$$I = 2 \int_0^{\pi/2} \sin^{2m-1} y \cdot \cos^{2n-1} y dy$$

$$I = \int_0^{\pi/2} \sin^{2m-1} y \cdot \cos^{2n-1} y dy = B(m,n) / 2$$

Formula (3)

We have,

$$I = \int_0^{\infty} x^{q-1} / (1+x) dx$$

Let

$$y = x / (1+x)$$

$$\text{Hence, } x = y / 1-y$$

$$, 1+x = 1 + (y / 1-y) = 1/(1-y)$$

$$\& dx = - [(1-y) - y(-1)] / (1-y)^2 \cdot dy = 1 / (1-y)^2 \cdot dy$$

whn $x = 0$, we have $y = 0$

when $x \rightarrow \infty$, we have $y = \lim_{x \rightarrow \infty} x / (1+x) = 1$

Thus,

$$I = \int_0^{\infty} [x^{q-1} / (1+x)] dx = \int_0^1 [(y / 1-y)^{q-1} / (1/(1-y))] \cdot 1 / (1-y)^2 \cdot dy$$

$$= \int_0^1 [y^{q-1} / (1-y)^{-q}] dy$$

$$= B(q, 1-q) = \Gamma(q) \Gamma(1-q)$$

Proving that $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = \int_0^{\infty} x^{1/2-1} e^{-x} dx = \int_0^{\infty} x^{-1/2} e^{-x} dx$$

$$\text{Let } y = x^{1/2}$$

$$x = y^2$$

$$dx = 2y dy$$

$$\Gamma(1/2) = \int_0^{\infty} y^{-1} e^{-y^2} 2y dy$$

$$= 2 \int_0^{\infty} e^{-y^2} dy$$

$$= 2 (\sqrt{\pi} / 2) = \sqrt{\pi}$$