* Application of Laplace Transform

Solution of Differential Equation Using Laplace Transform

Here, we use the derivative property as follows:

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Ex: Solve the D.E $x''(t) + 16x(t) = f(t)$

$x(0) = 0, x'(0) = 1$

Where $f(t) = \begin{cases} \cos 2t & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}$

Taking L.T for both sides

$$S^2 X(s) - S \cancel{x(0)}_{0} - \cancel{x'(0)} + 16 X(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 2t - \cos 2t \; u(t-\pi)\}$$

$$\cos 2t = \cos 2[(t-\pi)+\pi] = \cos(2(t-\pi) + 2\pi)$$

$$= \cos 2(t-\pi)$$

$$X(s)\left[s^2+16\right] = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-\pi s}.$$

$$X(s) = \frac{s}{(s^2+16)(s^2+4)} - \frac{s}{(s^2+16)(s^2+4)}e^{-\pi s}$$

$$X(s) = \frac{1}{16-4}\left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right) - \frac{1}{12}\left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right)e^{-\pi s}$$

$$x(t) = \frac{1}{12}\left[\cos 2t - \cos 4t\right] - \frac{1}{12}\left[\cos 2(t-\pi) - \cos 4(t-\pi)\right]u(t-\pi)$$

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Ex: Solve the following D.E

$$\frac{dy}{dt} - 3y = e^{2t} \qquad y(0) = 1$$

Soln $\mathcal{L}\{y'(t)\} = SY(S) - y(0)$

$$\mathcal{L}\{y'(t)\} = SY(S) - 1$$

$$SY(S) - 1 - 3Y(S) = \frac{1}{S-2}$$

$$Y(S)[S-3] = \frac{1}{S-2} + 1$$

$$Y(S) = \frac{1}{(S-2)(S-3)} + \frac{1}{(S-3)}$$

To get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(S)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-3)} + \frac{1}{s-3} \right\}.$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{-3+2}\left(\frac{1}{s-2} - \frac{1}{s-3} \right) + \frac{1}{s-3} \right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s-3} - \frac{1}{s-2} + \frac{1}{s-3} \right\}$$

$$y(t) = 2e^{3t} - e^{2t} \rightsquigarrow \text{Solution}$$

Ex

Solve the following D.E

$$y''(t) + 2y'(t) + y(t) = 3t\,e^{-t}, \quad y(0) = 4, \quad y'(0) = 2$$

Soln $\mathcal{L}\{y'(t)\} = SY(S) - y(0) = SY(S) - 4$

$$\mathcal{L}\{y''(t)\} = S^2Y(S) - Sy(0) - y'(0)$$

$$= S^2Y(S) - 4S - 2$$

$$\mathcal{L}\{t\,e^{-t}\} = \frac{1}{(S+1)^2}$$

$$\therefore S^2Y(S) - 4S - 2 + 2SY(S) - 8 + Y(S) = \frac{3}{(S+1)^2}$$

$$Y(S)\left[S^2 + 2S + 1\right] - 4S - 10 = \frac{3}{(S+1)^2}$$

$$Y(S)\left[(S+1)^2\right] = \frac{3}{(S+1)^2} + 10 + 4S$$

$$Y(S) = \frac{3}{(S+1)^4} + \frac{10}{(S+1)^2} + \frac{4S}{(S+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^4} + \frac{10}{(s+1)^2} + 4\frac{s}{(s+1)^2}\right\}$$

$$y(t) = 3e^{-t}\frac{t^3}{3!} + 10e^{-t}t + 4\mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2}\right\}$$

$$= 3e^{-t}\frac{t^3}{3!} + 10e^{-t}t + 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)} - \frac{1}{(s+1)^2}\right\}$$

$$y(t) = \frac{3}{3!}e^{-t}t^3 + 10e^{-t}t + 4e^{-t} - 4e^{-t}t$$

$$y(t) = \frac{3}{3!}t^3e^{-t} + 6te^{-t} + 4e^{-t}$$

Ex

→ Compute $y(\frac{\pi}{2})$, $y(3+\frac{\pi}{2})$ for the fn $y(t)$ which satisfies the boundary value problem $y''(t)+y(t)=(t-3)\,u(t-3)$

$$y(0)=y'(0)=0$$

Soln $S^2Y(S)-\cancel{S\,y(0)}-\cancel{y'(0)}+Y(S)$

$$=\frac{1}{S^2}e^{-3S}$$

$$Y(S)\left[S^2+1\right]=\frac{1}{S^2}e^{-3S}$$

$$Y(S)=\frac{1}{S^2(S^2+1)}e^{-3S}$$

$$Y(S)=\frac{1}{1-0}\left[\frac{1}{S^2}-\frac{1}{S^2+1}\right]e^{-3S}$$

$$y(t)=\mathcal{L}^{-1}\{Y(S)\}$$

$$y(t) = (t-3)u(t-3) - \sin(t-3)u(t-3)$$

$$y(t) = \left[(t-3) - \sin(t-3)\right]u(t-3)$$

$$y(t) = \begin{cases} 0 & 0 < t < 3 \\ (t-3) - \sin(t-3) & t > 3 \end{cases}$$

$$y\left(\frac{\pi}{2}\right) \rightsquigarrow \frac{\pi}{2} < 3 \quad \therefore y\left(\frac{\pi}{2}\right) = 0$$

$$y\left(\frac{\pi}{2}+3\right) \rightsquigarrow \frac{\pi}{2}+3 > 3$$

$$\therefore y\left(\frac{\pi}{2}+3\right) = \left(\frac{\pi}{2}+\cancel{3}-\cancel{3}\right) - \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}$$

$$= \frac{\pi}{2} - 1$$

[2] Solution of Integro - D.E

Ex Solve

$$f'(t) + 2f(t) + \int_0^t f(x)\,dx = 0 \quad , \; f(0) = 2$$

Sol^n Taking Laplace for both sides

$$sF(s) - \underbrace{f(0)}_{2} + 2F(s) + \frac{F(s)}{s} = 0$$

$$F(s)\left[s + 2 + \frac{1}{s}\right] = +2$$

$$F(s)\left[\frac{s^2+2s+1}{s}\right] = 2 \Rightarrow F(s) = \frac{2s}{s^2+2s+1}$$

$$F(s) = 2\,\frac{s}{(s+1)^2}$$

$$F(s) = 2\,\frac{(s+1)-1}{(s+1)^2} = 2\left[\frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$f(t) = 2\left(e^{-t} - t e^{-t}\right)$$

$\underline{Ex}$ $f'(t) = t + \int_0^t f(t-\lambda)\cos\lambda \, d\lambda$

$f(0) = 1$

$\underline{Soln}$

$$sF(s) - \cancel{f(0)}^{1} = \frac{1}{s^2} + F(s)\frac{s}{s^2+1}$$

$$F(s)\left[s - \frac{s}{s^2+1}\right] = \frac{1}{s^2} + 1$$

$$F(s)\left[\frac{s^3 + \cancel{s} - \cancel{s}}{(s^2+1)}\right] = \frac{1}{s^2} + 1$$

$$F(s) = \frac{s^2+1}{s^5} + \frac{s^2+1}{s^3} = \frac{1}{s^3} + \frac{1}{s^5} + \frac{1}{s} + \frac{1}{s^3}$$

$$F(s) = \frac{2}{s^3} + \frac{1}{s^5} + \frac{1}{s}$$

$$f(t) = 2\frac{t^2}{2!} + \frac{t^4}{4!} + 1$$