Chapter one

Homogenous 2nd order D.E.

Homogenous 2nd order D.E. with Constant Coeff.

Form

$$a.\frac{d^2y}{dx^2} + b.\frac{dy}{dx} + c.y = \text{Zero}$$

OR

$$a.y'' + b.y' + c.y = \text{Zero}$$

where

a, b & c $\rightarrow$ are constants

Consider the Solution is

$$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 . e^{mx}$$

Sub

$$\boxed{a.m^2 + bm + C = 0} \quad \text{C/C}$$

$\rightarrow$ Characteristic Equation

To Solve the C/C :

$$a.m^2 + b.m + c = 0$$

$$(m - \_\_).(m + \_\_) = 0$$

or Formula

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a.c}}{2a}$$

or Using Calc. Mode $\boxed{5} \rightarrow \boxed{3}$

$\therefore m_1 = \_\_$ & $m_2 = \_\_$

There are 3 Cases of Sol.

Case I $\quad b^2 - 4 \cdot a \cdot c > 0$

$\therefore$ We have two different real roots

$m_1 \quad m_2$

$$\therefore y = C_1 \cdot e^{m_1 \cdot x} + C_2 \cdot e^{m_2 \cdot x}$$

Ex: if $m = 2$ & $m = 3$

$$\boxed{\therefore y = C_1 \cdot e^{2x} + C_2 \cdot e^{3x}}$$

This is the Sol. of D.E.

Case II $\quad b^2 - 4a.c < 0$

$\therefore$ We have two imaginary roots

$$m_{1,2} = \alpha \pm i.\beta$$

$$\therefore y = e^{\alpha.x}\left(C_1.\cos\beta x + C_2.\sin\beta x\right)$$

Ex: if $\quad m_{1,2} = \underbrace{2}_{\alpha} \pm i \underbrace{3}_{\beta}$

$$\boxed{\therefore y = e^{2.x}\left(C_1.\cos(3x) + C_2.\sin(3x)\right)}$$

Case III $\quad b^2 - 4a \cdot c = 0$

$\therefore$ We have two equal real roots

$$m_1 = m_2 = m$$

$$\therefore \; y = (c_1 \cdot x + c_2 \cdot) \cdot e^{mx}$$

<u>Ex</u>: if $m = 3$ & $m = 3$

$$\therefore \; \boxed{y = (c_1 \cdot x + c_2) \cdot e^{3x}}$$

Ex : (1) Solve the following D.E.

(1) $y'' + 5 \cdot y' + 6 \cdot y = 0$

Sol.

Assume

$$y = e^{m \cdot x} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$$

Sub.

$$(m^2 + 5 \cdot m + 6) \cdot \cancel{e^{mx}} = 0$$

$\neq 0$

C/C

$$m^2 + 5 \cdot m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$\boxed{m = -2} \quad \& \quad \boxed{m = -3}$$

Case I

$$\therefore \; y = C_1 \cdot e^{-2x} + C_2 \cdot e^{-3x}$$

② $y'' + 4 \cdot y' + 13 \cdot y = 0$

Sol.

Assume

$y = e^{mx} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 + 4 \cdot m + 13) \cdot \cancel{e^{mx}} = 0$

$\neq 0$

c/c

$m^2 + 4m + 13 = 0$

$\therefore m_{1,2} = \dfrac{-4 \pm \sqrt{16 - 4 * 1 * 13}}{2 * 1} = \dfrac{-4 \pm \sqrt{-36}}{2}$

$\therefore m_{1,2} = \dfrac{-4 \pm i \cdot 6}{2} = -2 \pm i\, 3$

α $\qquad$ β

$\therefore y = e^{-2x} \left(C_1 \cdot \cos 3x + C_2 \cdot \sin 3x \right)$

③ $y'' - 4 \cdot y' + 4 \cdot y = 0$

Sol

Assume

$y = e^{m \cdot x} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 - 4m + 4) \cdot \cancel{e^{mx}} = 0$

C/C

$m^2 - 4 \cdot m + 4 = 0$

$(m - 2)(m - 2) = 0$ Case III

$\boxed{m=2}$ & $\boxed{m=2}$

$\therefore \; y = (c_1 \cdot x + c_2) \cdot e^{2x}$

(5) $y'' - 3 \cdot y' = 0$

Soln

Assume

$y = e^{mx} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 - 3m) \cdot \cancel{e^{mx}} = 0$

C/C

$m^2 - 3m = 0$

$m \cdot (m - 3) = 0$

$\boxed{m = 0}$ & $\boxed{m = 3}$

Case I

$\therefore y = c_1 \cdot \cancel{e^{0x}}_1 + c_2 \cdot e^{3x}$

$\therefore y = c_1 + c_2 \cdot e^{3x}$

(6) $y'' + 16 \cdot y = 0$

Soln

Assume

$y = e^{mx} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub)

$(m^2 + 16) \cdot \cancel{e^{mx}} = 0$

c/c $\quad m^2 + 16 = 0$

$m^2 = -16$

$m = \sqrt{-16} = \pm i \cdot 4 = \underset{\alpha}{0} \pm i \underset{\beta}{4}$

$\therefore \; y = \cancel{e^{0x}} \cdot [c_1 \cdot \cos 4x + c_2 \cdot \sin 4x]$

$\therefore \; y = c_1 \cdot \cos 4x + c_2 \cdot \sin 4x$

(8) $y''' - 4 \cdot y'' + 3 \cdot y' = 0$

Soln

Assume

$y = e^{mx} \rightarrow y' = m \cdot e^{mx} \rightarrow y'' = m^2 \cdot e^{mx} \rightarrow y''' = m^3 \cdot e^{mx}$

Sub

$(m^3 - 4 \cdot m^2 + 3m) \cdot \cancel{e^{mx}} = 0$

C/C $\quad m^3 - 4 \cdot m^2 + 3 \cdot m = 0$

$m \cdot (m^2 - 4 \cdot m + 3) = 0$ $\quad$ Case I

$m \cdot (m - 1)(m - 3) = 0$

$\therefore \boxed{m = 0}, \boxed{m = 1}, \boxed{m = 3}$

$\therefore y = c_1 \cdot \cancel{e^{0x}}_{1} + c_2 \cdot e^{1 \cdot x} + c_3 \cdot e^{3 \cdot x}$

(10) $y''' - 6 \cdot y'' + 12 \cdot y' - 8 \cdot y = 0$

Sol.

Assume

$y = e^{mx} \rightarrow y' = m \cdot e^{mx} \rightarrow y'' = \checkmark \rightarrow y''' = \checkmark$

Sub

$(m^3 - 6 \cdot m^2 + 12 \cdot m - 8) \cdot \cancel{e^{mx}} = 0$

ممكن نستخدم Calc. Mode $\rightarrow$ $\boxed{5}$ $\rightarrow$ $\boxed{4}$

C/C $\quad m^3 - 6m^2 + 12 \cdot m - 8 = 0$

$m^3 - 8 - 6m^2 + 12m = 0$

$(m-2)(m^2 + 2m + 4) - 6 \cdot m \cdot (m-2) = 0$

$(m-2) \cdot (m^2 - 4m + 4) = 0$

$(m-2) \cdot (m-2) \cdot (m-2) = 0$

$\boxed{m = 2}$, $\boxed{m = 2}$, $\boxed{m = 2}$

$$\therefore y = (c_1 x^2 + c_2 x + c_3) \cdot e^{2x}$$

(11) $\frac{d^4y}{dx^4} - 5 \cdot \frac{d^2y}{dx^2} + 4 \cdot y = 0$

Soln

Assume

$y = e^{mx} \to y' = m \cdot e^{mx} \to y'' \longrightarrow$

Sub

$(m^4 - 5 \cdot m^2 + 4) \cdot \cancel{e^{mx}} = 0$

c/c $\quad m^4 - 5 \cdot m^2 + 4 = 0$

$(m^2 - 1) \cdot (m^2 - 4) = 0$

$(m-1)(m+1) \cdot (m-2) \cdot (m+2) = 0$

$\therefore \boxed{m = 1}, \boxed{m = -1}, \boxed{m = 2}, \boxed{m = -2}$

$\therefore y = c_1 \cdot e^{x} + c_2 \cdot e^{-x} + c_3 \cdot e^{2x} + c_4 \cdot e^{-2x}$

Exercise

Solve the following Homog. D.E

① $y'' + y' - 2y = 0$

② $y'' - 3y' + 2y = 0$

③ $y'' + y' = 0$

④ $y'' + 4y' + 5y = 0$

⑤ $y'' + 4y = 0$

⑥ $y'' - y = 0$

⑦ $y''' - 3y'' + 3y' - y = 0$

⑧ $y^{(4)} - 4y'' = 0$