Chapter one

Differential Equations

A ***Differential Equation*** is an equation that contains one or more derivatives of a differentiable function. An equation with partial derivatives is called a ***Partial Differential Equation***. While, an equation with ordinary derivatives, that is, derivatives of a function of a single variable, is called an ***Ordinary Differential Equation***.

The ***order*** of a differential equation is the order of the equation's highest order derivative. A differential equation is ***linear*** if it can be put in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

The ***degree*** of a differential equation is the power (exponent) of the equation's highest order derivative.

<u>Example</u>

First order, first degree, linear $\qquad \frac{dy}{dx} = 5y, \qquad 3\frac{dy}{dx} - \sin x = 0$

Third order, second degree, nonlinear $\qquad \left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx^2}\right)^5 - \frac{dy}{dx} = e^x$

Solution of First Order Differential Equations

1) Separable Equations

A first order differential equations is separable if it can be put in the form

$$M(x)dx + N(y)dy = 0$$

Steps for Solving a Separable First Order Differential Equation

i. Write the equation in the form $M(x)dx + N(y)dy = 0$.

ii. Integrate M with respect to x and N with respect to y to obtain an equation that relates y and x.

Example

Solve the following differential equations

$$\text{(a) } \frac{dy}{dx} = (1+y^2)e^x, \qquad \text{(b) } \frac{dy}{dx} = \frac{x(2\ln x + 1)}{\sin y + y\cos y}$$

Solution

(a) $\frac{dy}{dx} = (1+y^2)e^x \quad \Rightarrow \quad e^x dx - \frac{1}{1+y^2}dy = 0$

$$\int e^x dx - \int \frac{1}{1+y^2} dy = C \quad \Rightarrow \quad e^x - \tan^{-1} y = C$$

$$\tan^{-1} y = e^x - C \quad \Rightarrow \quad y = \tan(e^x - C)$$

(b) $\frac{dy}{dx} = \frac{x(2\ln x + 1)}{\sin y + y\cos y} \quad \Rightarrow \quad (\sin y + y\cos y)dy = x(2\ln x + 1)dx$

$$\int (\sin y + y\cos y)dy - \int x(2\ln x + 1)dx = C$$

$$\int \sin(y)dy + \int y\cos(y)dy - 2\int x\ln(x)dx - \int xdx = C$$

$$-\cos(y) + \left[y\sin y - \int \sin(y)dy\right] - 2\left[\ln(x)\times\frac{x^2}{2} - \int \frac{x^2}{2}\times\frac{1}{x}dx\right] - \frac{x^2}{2} = C$$

$$-\cos(y) + y\sin y + \cos y - x^2\ln x + \frac{x^2}{2} - \frac{x^2}{2} = C$$

$$y\sin y - x^2\ln x = C.$$

Notes

1) $f_1(x)g_1(y)dy + f_2(x)g_2(y)dx = 0$ *Separable*

2) $\dfrac{f_1(x)}{g_1(y)}dy + \dfrac{f_2(x)}{g_2(y)}dx = 0$ *Separable*

3) $[f_1(x) \pm g_1(y)]dy + [f_2(x) \pm g_2(y)]dx$ *Not Separable*

Example

$$f_1(x) = x, \quad f_2(x) = \sin(x), \quad g_1(y) = y, \quad g_2(y) = \tan(y)$$

1) $xydy + \sin(x)\tan(y)dx = 0 \quad \Rightarrow \quad xydy = -\sin(x)\tan(y)dx$

$\dfrac{y}{\tan(y)}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

2) $\dfrac{x}{y}dy + \dfrac{\sin(x)}{\tan(y)}dx = 0 \quad \Rightarrow \quad \dfrac{x}{y}dy = -\dfrac{\sin(x)}{\tan(y)}dx$

$\dfrac{\tan(y)}{y}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

3) $(x+y)dy + (\sin(x) + \tan(y))dx = 0$

$(x+y)dy = -(\sin(x) + \tan(y))dx$ *Not Separable*

Special Type of Separable Equations

If $\frac{dy}{dx} = f(ax+by+c)$; then let $z = ax+by+c$ and the resultant equation may be reduced to a separable equation.

Example

Solve the differential equation $\frac{dy}{dx} = \tan^2(x+y)$

Solution

$$z = x+y \quad \Rightarrow \quad \frac{dz}{dx} = 1+\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \Rightarrow \quad \frac{dz}{dx} - 1 = \tan^2(z)$$

$$\frac{dz}{dx} = \tan^2(z)+1 \quad \Rightarrow \quad \frac{dz}{dx} = \sec^2(z)$$

$$\frac{dz}{\sec^2(z)} = dx \quad \Rightarrow \quad \cos^2(z)dz = dx$$

$$\int \cos^2(z)dz - \int dx = C \quad \Rightarrow \quad \int \frac{1+\cos(2z)}{2}dz - \int dx = C$$

$$\frac{1}{2}z + \frac{1}{4}\sin(2z) - x = C$$

While $z = x+y$, then the solution is

$$\frac{1}{2}(x+y) + \frac{1}{4}\sin(2(x+y)) - x = C$$

Homogeneous Function

If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then $f(x, y)$ is homogeneous function and n represents the degree of the homogeneous function.

Example

For the function $f(x, y) = x^2 + y^2$ then

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$

$$= \lambda^2 x^2 + \lambda^2 y^2$$

$$= \lambda^2 (x^2 + y^2) = \lambda^2 f(x, y)$$

So, the function $f(x, y)$ is homogeneous with degree 2.

Example

For the function $f(x, y) = x + y^2$ then

$$f(\lambda x, \lambda y) = \lambda x + (\lambda y)^2$$

$$= \lambda x + \lambda^2 y^2$$

$$= \lambda (x + \lambda y^2)$$

So, the function $f(x, y)$ is not homogeneous.

Example

$f(x, y) = x^2 + y^2 + 5$ (Non-homogeneous)

$f(x, y) = x^3 + xy + x$ (Non-homogeneous)

$f(x, y) = \cos(xy)$ (Non-homogeneous)

$f(x, y) = \cos(x^2 \pm y^2)$ (Non-homogeneous)

$f(x, y) = \cos\left(\frac{x}{y}\right)$ (Homogeneous)

$f(x, y) = \cos\left(\frac{x^2}{y}\right)$ (Non-homogeneous)

Homogeneous Equations

The differential equation $M(x, y)dx + N(x, y)dy$ is homogeneous if M and N are homogeneous functions of the same degree.

Example

1) $(x^2 + y^2)dx + xydy = 0$

This is homogeneous because M and N are both homogeneous with degree 2.

2) $(x^3 + y^3)dx + xydy = 0$

This is not homogeneous because M is homogeneous with degree 3 while N is homogeneous with degree 2.

3) $xdx + (x^2 + y)dy = 0$

This is not homogeneous because N is not homogeneous.

Solution of Homogeneous Equations

A homogeneous first order differential equation can be put in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

This equation can be changed into separable equation with the substitutions

$$v = \frac{y}{x} \quad \Rightarrow \quad y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then becomes
$$v + x\frac{dv}{dx} = F(v)$$

which can be rearranged algebraically to give

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

with the variables now separated, the equation can now be solved by integrating with respect to x and v. We can then return to x and y by substituting $v = y/x$.

Example

Find the solution of the differential equation

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$$

that satisfies the condition $y(1) = 1$.

Solution

Dividing the numerator and denominator of the right-hand side by x^2 gives

$$\frac{dy}{dx} = -\frac{1+(y/x)^2}{2(y/x)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1+v^2}{2v} = F(v)$$

$$v - F(v) = v + \frac{1+v^2}{2v} = \frac{2v^2+1+v^2}{2v} = \frac{3v^2+1}{2v}$$

$$\frac{dx}{x} + \frac{dv}{v-F(v)} = 0 \quad \Rightarrow \quad \frac{dx}{x} + \frac{2vdv}{3v^2+1} = 0$$

The solution of this equation can be written as

$$\int \frac{dx}{x} + \int \frac{2vdv}{3v^2+1} = C \quad \Rightarrow \quad \ln x + \frac{1}{3}\ln(1+3v^2) = C$$

$$3\ln x + \ln(1+3v^2) = 3C \quad \Rightarrow \quad \ln x^3 + \ln(1+3v^2) = 3C$$

$$e^{\ln x^3 + \ln(1+3v^2)} = e^{3C} \quad \Rightarrow \quad e^{\ln x^3} \times e^{\ln(1+3v^2)} = e^{3C}$$

$$x^3(1+3v^2) = C' \quad \Rightarrow \quad x^3\left(1+3\frac{y^2}{x^2}\right) = C'$$

$$x^3 + 3xy^2 = C'$$

The condition is that when $x = 1$ then $y = 1$ and the constant C' can be found

$$(1)^3 + 3(1)(1)^2 = C' \qquad \Rightarrow \qquad C' = 4$$

The final solution is $x^3 + 3xy^2 = 4$.

Reducible to Homogeneous

If the differential equation has the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case 1: if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then $z = a_1x + b_1y$

Case 2: if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then intersect the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to find the intersection point (h, k) and let

$$x = X + h \quad \Rightarrow \quad dx = dX, \text{ and } \quad y = Y + k \quad \Rightarrow \quad dy = dY$$

Example

Solve the differential equation $\frac{dx}{dy}=\frac{4x+6y+5}{3y+2x+4}$

Solution

$$\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}$$

$$a_1=2,\quad a_2=4 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}$$

$$b_1=3,\quad b_2=6 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$$

So $\quad \frac{a_1}{a_2}=\frac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 1***

Let $\quad z=2x+3y \qquad \Rightarrow \qquad \frac{dz}{dx}=2+3\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right) \quad \Rightarrow \quad \frac{1}{3}\frac{dz}{dx}-\frac{2}{3}=\frac{z+4}{2z+5} \quad \Rightarrow \quad \frac{dz}{dx}=\frac{3z+12}{2z+5}+2$$

$$\frac{dz}{dx}=\frac{3z+12+4z+10}{2z+5} \quad \Rightarrow \quad \frac{dz}{dx}=\frac{7z+22}{2z+5} \quad \Rightarrow \quad \frac{2z+5}{7z+22}dz=dx$$

$$\int\frac{2z+5}{7z+22}dz-\int dx=C \quad \Rightarrow \quad \int\left(\frac{2}{7}-\frac{9}{7}\times\frac{1}{7z+22}\right)dz-\int dx=C$$

$$\int\frac{2}{7}dz-\int\frac{9}{7\times7}\times\frac{7}{7z+22}dz-\int dx=C$$

$$\frac{2}{7}z-\frac{9}{49}\times\ln(7z+22)-x=C$$

$$\frac{2}{7}(2x+3y)-\frac{9}{49}\times\ln\big(7(2x+3y)+22\big)-x=C$$

Example

Solve the differential equation $(2x+y-3)dy=(x+2y-3)dx$

Solution

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

$$a_1=1,\quad a_2=2 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{1}{2}$$

$$b_1=2,\quad b_2=1 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=2$$

So $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ $\qquad \Rightarrow \qquad$ ***Case 2***

$$\begin{array}{llll} x & +2y & -3 & =0 \quad \ldots\ldots \quad (1)\\ 2x & +y & -3 & =0 \quad \ldots\ldots \quad (2)\\ \hline +2x & +4y & -6 & =0\\ \mp 2x & \mp y & \pm 3 & =0\\ \hline & 3y & -3 & =0 \end{array}$$

$$\Rightarrow \quad y=1$$

Substituting into (2), we get

$$2x+1-3=0 \quad \Rightarrow \quad x=1$$

The intersection point $(h,k)=(1,1)$.

Let $\quad x=X+1 \quad \Rightarrow \quad dx=dX$

$$y=Y+1 \quad \Rightarrow \quad dy=dY$$

$$\frac{dY}{dX}=\frac{(X+1)+2(Y+1)-3}{2(X+1)+(Y+1)-3}$$

$$=\frac{X+1+2Y+2-3}{2X+2+Y+1-3} \qquad \Rightarrow \qquad \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

$$\frac{dY}{dX}=\frac{1+2\frac{Y}{X}}{2+\frac{Y}{X}}$$

Let $v=\frac{Y}{X}$ $\Rightarrow$ $\frac{dY}{dX}=\frac{1+2v}{2+v}=F(v)$

$$\frac{dX}{X}+\frac{dv}{v-F(v)}=0$$

$$v-F(v)=v-\frac{1+2v}{2+v}$$

$$=\frac{2v+v^2-1-2v}{2+v}=\frac{v^2-1}{2+v}$$

$$\frac{dX}{X}+\frac{2+v}{v^2-1}dv=0 \qquad \Rightarrow \qquad \ln X+\int\frac{2+v}{v^2-1}dv=C$$

$$\frac{2+v}{v^2-1}=\frac{A}{v+1}+\frac{B}{v-1}=\frac{A(v-1)+B(v+1)}{v^2-1}$$

$$2+v=A(v-1)+B(v+1)$$

$$A=\frac{-1}{2} \qquad B=\frac{3}{2}$$

$$\ln X+\int\frac{-1/2}{v+1}dv+\int\frac{3/2}{v-1}dv=C$$

$$\ln X-\frac{1}{2}\ln(v+1)+\frac{3}{2}\ln(v-1)=C$$

$$\ln X-\frac{1}{2}\ln(\frac{Y}{X}+1)+\frac{3}{2}\ln(\frac{Y}{X}-1)=C$$

$$\ln(x-1)-\frac{1}{2}\ln(\frac{(y-1)}{(x-1)}+1)+\frac{3}{2}\ln(\frac{(y-1)}{(x-1)}-1)=C$$

2) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

where $$\rho(x)=e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(x)$.

iii. Find the integrating factor $\rho(x)=e^{\int P(x)dx}$.

iv. Find y using the following equation

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example

Solve the equation $x\frac{dy}{dx}-3y=x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions*** P ***and*** Q. To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx}-\frac{3}{x}y=x \qquad \Rightarrow \qquad P(x)=-\frac{3}{x}, \qquad Q(x)=x.$$