Chapter four

Laplace Transform

Definition

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)}$$

$s > 0$

This transformation transforms from Domain to another Domain

$$\begin{tikzcd}
\begin{array}{c}\text{Time} \\ \text{Domain} \\ T\end{array} \arrow[r, bend left, "L.T"] & \begin{array}{c}\text{Frequency} \\ \text{Domain} \\ S\end{array} \arrow[l, bend left, "(L.T)^{-1}"]
\end{tikzcd}$$

Laplace Transform of Elementary f^{n} :-

$\boxed{1}$ $f(t) = K \longrightarrow$ Constant

$$F(s) = \int_0^\infty e^{-st} K \, dt = K\left(\frac{-1}{s}\right)\left(e^{-st}\right)_0^\infty$$

$$= -\frac{K}{s}\left(e^{-s(\infty)} - 1\right)$$

$$s > 0 \quad \therefore \; e^{-(s)\infty} = 0$$

$$F(s) = -\frac{K}{s}(0 - 1) = \frac{K}{s}$$

$$\mathcal{L}\{K\} = \frac{K}{s} \quad s > 0$$

$\boxed{2}$ $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \frac{-1}{s-a}\left(e^{-(s-a)t}\right)_0^\infty = \frac{-1}{s-a}(0 - 1) \quad s > a$$

$$= \frac{1}{s-a}, \; s > a$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \; s > a$$

3 $f(t) = \cosh at$ $\qquad \cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$F(s) = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} + e^{-(s+a)t} \right) dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$s > a \qquad\qquad s > -a$$

$$= \frac{1}{2} \quad \frac{s + \cancel{a} + s - \cancel{a}}{s^2 - a^2}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}, \quad s > |a|$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}, \quad s > |a|$$ بالمثل،

$\boxed{4}$ $f(t) = \sin at$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \frac{-1}{a} \int_0^\infty \underbrace{e^{-st}}_{u} \, d \, \underbrace{\cos at}_{v} = -\frac{1}{a} \left(e^{-st} \cos at \right)_0^\infty + \frac{1}{a} \int_0^\infty \cos at \, d e^{-st}$$

$$= -\frac{1}{a} (0 - 1) + \frac{1}{a} \int_0^\infty (-s) \, e^{-st} \cos at \, dt$$

$$= \frac{1}{a} - \frac{s}{a^2} \int_0^\infty \underbrace{e^{-st}}_{u} \, d \, \underbrace{\sin at}_{v}$$

$$= \frac{1}{a} - \frac{s}{a^2} \left(e^{-st} \underbrace{\sin at}_{\to 0} \right)_0^\infty + \frac{s}{a^2} \int_0^\infty \sin at \, d e^{-st}$$

$$= \frac{1}{a} + \frac{s}{a^2} \int_0^\infty (-s) \, e^{-st} \sin at \, dt$$

I

$$I = \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt$$

$$I \left(1 + \frac{s^2}{a^2} \right) = \frac{1}{a} \quad \therefore I \left(\frac{a^2 + s^2}{a^{\cancel{2}}} \right) = \frac{1}{\cancel{a}}$$

$$\therefore I = \frac{a}{s^2 + a^2}$$

$$\therefore \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

وبالمثل

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Summary

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

Time domain $f(t)$	Laplace s-domain $F(s)$	Region of Convergence
K	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	عدد صحيح موجب n +ve integer $s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
e^{-at}	$\frac{1}{s+a}$	$s > -a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$

Note That

1) $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

2) $\mathcal{L}\{K f(t)\} = K \mathcal{L}\{f(t)\}$

3) $\mathcal{L}\{f(t) g(t)\} \neq \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$

لا يساوي

هام جداً

* Take Care

* $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

* $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

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* $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

* $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

* $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

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* $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

* $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

* $\cos(x \pm y)$
$= \cos x \cos y \mp \sin x \sin y$

* $\sin(x \pm y)$
$= \sin x \cos y \pm \sin y \cos x$

Examples

Find Laplace transform for

1. $f(t) = e^{-3t} + t^5 + 7t^2 - 3\sin 2t$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+3} + \frac{5!}{s^6} + 7\frac{2!}{s^3} - 3 * \frac{2}{s^2+4}$$

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2. $f(t) = 3\cosh 2t - 7\sin 3t + 4 - t^3$

$$\mathcal{L}\{f(t)\} = 3\frac{s}{s^2-4} - 7 * \frac{3}{s^2+9} + 4/s - \frac{3!}{s^4}$$

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3. $f(t) = (t^2+3)^2$

$$f(t) = t^4 + 6t^2 + 9$$

$$\mathcal{L}\{f(t)\} = \frac{4!}{s^5} + 6\frac{2!}{s^3} + 9/s$$

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4. $f(t) = (e^{-2t} + 1)^2$

$$f(t) = e^{-4t} + 2e^{-2t} + 1$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+4} + 2\frac{1}{s+2} + 1/s$$

5 $f(t) = \cos 3t + \sin 2t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 3t + \sin 2t\}$$

$$= \frac{s}{s^2+9} + \frac{2}{s^2+4}$$

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6 $f(t) = \cos^2 3t$

$$\cos 6t = 2\cos^2 3t - 1$$

$$\cos^2 3t = \left(\frac{1+\cos 6t}{2}\right)$$

$$\therefore f(t) = \frac{1}{2}(1+\cos 6t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

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7 $f(t) = \cos 3t \sin t = \sin t \cos 3t$

We Know $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$$f(t) = \frac{1}{2}\left[\sin(4t) + \sin(-2t)\right] = \frac{1}{2}\left[\sin(4t) - \sin(2t)\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{4}{s^2+16} - \frac{2}{s^2+4}\right]$$

$\boxed{8}$ $f(t) = \sin^3 t$

$$f(t) = \sin t \, \sin^2 t$$

$$= \sin t \left[\tfrac{1}{2}(1 - \cos 2t)\right]$$

$(\sin t \cos 2t)$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos 2t \sin t$$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{2}\left(\tfrac{1}{2}(\sin 3t - \sin t)\right)$$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{4}\sin 3t + \tfrac{1}{4}\sin t$$

$$f(t) = \tfrac{3}{4}\sin t - \tfrac{1}{4}\sin 3t$$

$$\mathcal{L}\{f(t)\} = \frac{3}{4}\,\frac{1}{s^2+1} - \frac{1}{4}\,\frac{3}{s^2+9}$$

$\boxed{9}$ $f(t) = \sin(3t+2)$

$$f(t) = \sin 3t \cos(2) + \cos(3t)\sin(2)$$

We know that $\sin(x+y) = \sin x \cos y + \sin y \cos x$

$$\mathcal{L}\{f(t)\} = \cos(2)\,\frac{3}{s^2+9} + \sin(2)\,\frac{s}{s^2+9}$$

ثوابت