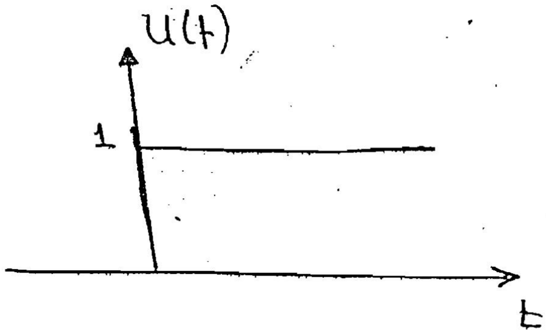
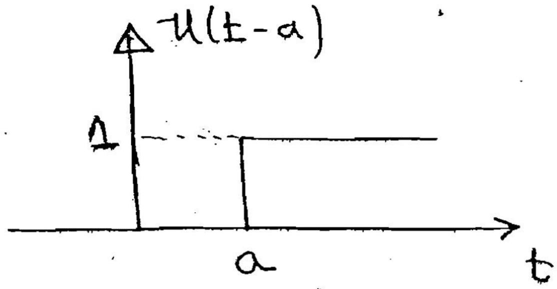
* The unit step function

* $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



* $u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$



* Prove That $\mathcal{L}\{u(t-a)\} = \frac{1}{s} e^{-as}$

Soln $\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$

$$= \int_a^a (0) + \int_a^\infty e^{-st}\, dt = -\frac{1}{s}\left[e^{-st}\right]_a^\infty$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right] = \frac{1}{s} e^{-as}$$

$$\therefore \boxed{\mathcal{L}\{u(t-a)\} = \frac{1}{s} e^{-as}}$$

* Special case for $a = 0$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

Expressing any Function as a unit Step Function

كيف نحول اى داله بداله unit step

* If $f(t) = g(t) \quad t < a$

$\quad = h(t) \quad t > a$

$$\boxed{f(t) = g(t) + [h(t) - g(t)]\, u(t-a)}$$

* If $f(t) = g_1(t) \quad t > a$

$\quad = g_2(t) \quad a < t < b$

$\quad = g_3(t) \quad t > b$

$$\boxed{f(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t-a) + [g_3(t) - g_2(t)]\, u(t-b)}$$

Ex:- Express the following functions as a unit step

1) $f(t) = 2 \quad 0 < t < 3$

$\quad = -2 \quad t > 3$

2) $f(t) = 0 \quad 0 < t < \frac{3\pi}{2}$

$\quad = \sin t \quad t > \frac{3\pi}{2}$

3) $f(t) = t^2 \quad 0 < t < 3$

$\quad = 9 \quad 3 < t < 5$

$\quad = 0 \quad t > 5$

Soln

1) $f(t) = 2 \quad 0 < t < 3$

$= -2 \quad t > 3$

$f(t) = 2 + (-2-2)u(t-3)$

$f(t) = 2 - 4u(t-3)$

2) $f(t) = 0 \quad 0 < t \leq 3\pi/2$

$= \sin t \quad t > 3\pi/2$

$f(t) = 0 + (\sin t - 0)u(t - \frac{3\pi}{2})$

$f(t) = \sin t \, u(t - \frac{3\pi}{2})$

3) $f(t) = t^2 \quad 0 < t \leq 3$

$= 9 \quad 3 < t < 5$

$= 0 \quad t > 5$

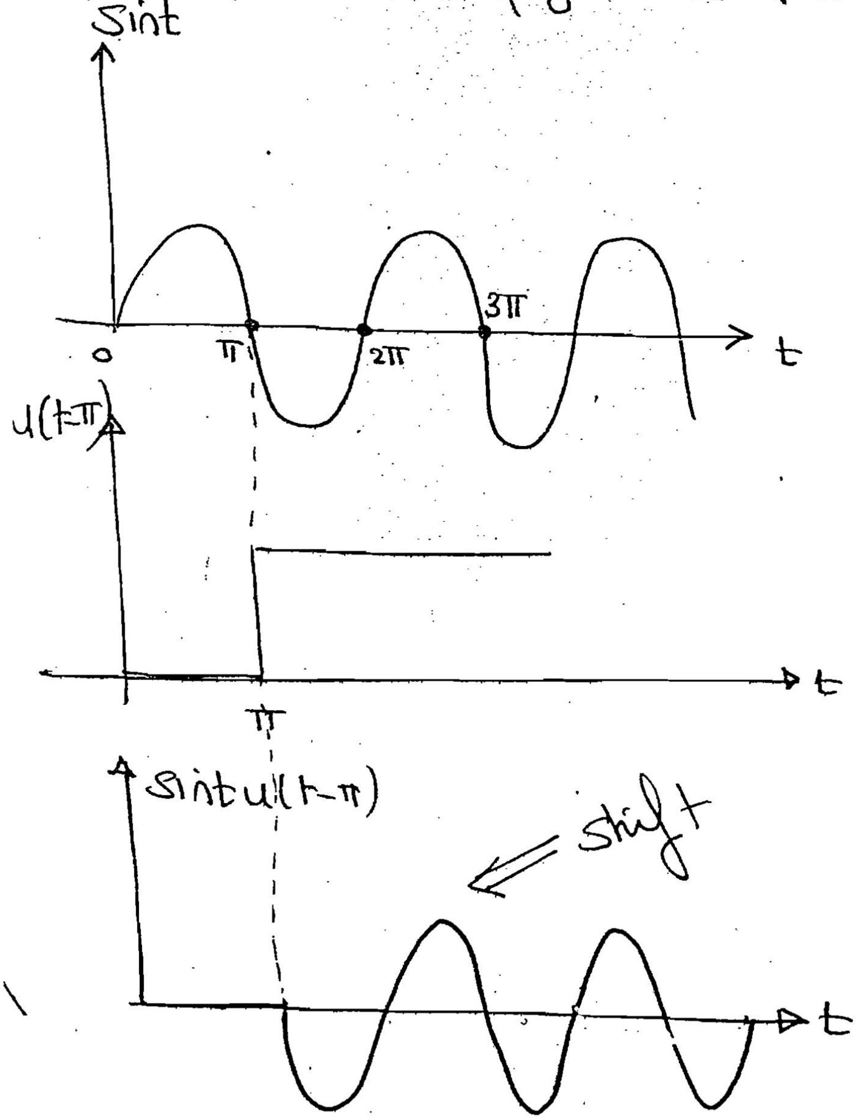
$f(t) = t^2 + (9 - t^2)u(t-3) + (0-9)u(t-5)$

$f(t) = t^2 + (9-t^2)u(t-3) - 9u(t-5)$

* Ex:- Draw the given $\underline{fn}$

$$f(t) = \sin t \, u(t-\pi)$$

$$u(t-\pi) = \begin{cases} 1 & t > \pi \\ 0 & t < \pi \end{cases}$$



6 2^{nd} Shift Theorm

If $\mathcal{L}\{f(t)\} = F(s)$, then prove

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = F(s)\,e^{-as}$$

Soln

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$u(t-a) = \begin{cases} 1 & t>a \\ 0 & t<a \end{cases}$$

$$\therefore \mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^a \cancel{(0)}\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

put $t-a = x$ $\quad dt = dx \quad t = a+x \quad$ at $t=a$ $\boxed{x=0}$

at $t=\infty$ $\boxed{x=\infty}$

$$\mathcal{L}\{\;\} = \int_0^\infty e^{-s(x+a)}\,f(x)\,dx$$

$$= \int_0^\infty e^{-sx}\,e^{-sa}\,f(x)\,dx$$

$$= e^{-as}\int_0^\infty e^{-sx} f(x)\,dx = e^{-as}F(s)$$

$$\boxed{\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)}$$

* Steps of Solution of 2nd shift

Ex

$$\mathcal{L}\{(t-3)\,u(t-3)\}$$

① التأكد ان كل t عباره عن (t-3) وهي موجوده
داخل (u)

$$\mathcal{L}\{(t\cancel{-3)}\,u(t\cancel{-3)})\} = \frac{1}{s^2}\,e^{-3s}$$

Ex: $\mathcal{L}\{(t-1)^3\,u(t-1)\}$

$\mathcal{L}\{t^3\}$

Soln $$\mathcal{L}\{(t\cancel{-1})^3\,u(t\cancel{-1})\} = \frac{3!}{s^4}\,e^{-s}$$

$$\because\ \mathcal{L}\{e^{3t-3}\,u(t-1)\}$$

$\mathcal{L}\{e^{3t}\}$

$$\therefore\ \mathcal{L}\{e^{3(t\cancel{-1})}\,u(t\cancel{-1})\} = \frac{1}{s-3}\,e^{-s}$$

Ex: $\mathcal{L}\{e^{2-t}\, u(t-2)\}$

Soln $\mathcal{L}\{e^{-(t-2)}\, u(t-2)\} = \frac{1}{S+1}\, e^{-2S}$

$\mathcal{L}\{e^{-t}\}$

Ex: $\mathcal{L}\{\cos 2t\; u(t-\pi)\}$

Soln لازم كل t تكون على صورة $t-\pi$

$\therefore \cos 2t = \cos 2(t-\pi+\pi)$

$= \cos[(2t-2\pi)+2\pi]$

$= \cos(2t-2\pi)\underbrace{\cos 2\pi}_{1} - \underbrace{\sin 2\pi}_{0}\sin(2t-2\pi)$

$= \cos 2(t-\pi)$

$\mathcal{L}\{\cos 2t\, u(t-\pi)\} = \mathcal{L}\{\cos 2(t-\cancel{\pi})\, u(t-\cancel{\pi})\}$

$= \frac{S}{S^2+4}\, e^{-\pi S}$

$\mathcal{L}\{\cos 2t\}$

Ex: $\mathcal{L}\{\sin t \; u(t-2\pi)\}$

Soln $\quad \sin t = \sin[(t-2\pi)+2\pi]$

$$= \sin(t-2\pi)\underbrace{\cos 2\pi}_{1} + \cos(t-2\pi)\underbrace{\sin 2\pi}_{0}$$

$$= \sin(t-2\pi)$$

$$\therefore \mathcal{L}\{\sin t \; u(t-2\pi)\} = \mathcal{L}\{\sin(t-2\pi)\,u(t-2\pi)\}$$

$$= \mathcal{L}\{\sin t\}\; e^{-2\pi s}$$

$$= \frac{1}{s^2+1}\;\bar{e}^{2\pi s}$$

-x: $\mathcal{L}\{\sin t \; u(t-\pi/2)\}$

ln $\quad \sin t = \sin[(t-\pi/2)+\pi/2]$

$$= \sin(t-\pi/2)\underbrace{\cos \pi/2}_{0} + \underbrace{\sin \pi/2}_{1}\cos(t-\pi/2)$$

$$\text{int } u(t-\pi/2)\} = \mathcal{L}\{\cos(t-\pi/2)\,u(t-\pi/2)\}$$

$$= \frac{s}{s^2+1}\;\bar{e}^{\pi/2\, s}$$

Ex: $\mathcal{L}\{(t-1)^3 e^{t-1} u(t-1)\}$

Soln $\mathcal{L}\{e^{t-\cancel{1}}(t-\cancel{1})^3 u(t-\cancel{1})\}$

$= \mathcal{L}\{e^{t} t^3\} e^{-s}$

$= \dfrac{3!}{(s-1)^4} e^{-s}$

Ex: $\mathcal{L}\{t\,u(t-3)\}$

Soln $\mathcal{L}\{t\,u(t-3)\} = \mathcal{L}\{[t-3+3]\,u(t-3)]$

$= \mathcal{L}\{(t-\cancel{3})u(t-\cancel{3}) + 3\,u(t-\cancel{3})\}$

$= \dfrac{1}{s^2} e^{-3s} + 3\dfrac{1}{s} e^{-3s}$

Ex:- $\mathcal{L}\{t^2 u(t-2)\}$

Soln $\quad t^2 = [(t-2)+2]^2 = (t-2)^2 + 4(t-2) + 4$

$\mathcal{L}\{t^2 u(t-2)\} = \mathcal{L}\{[(t-2)^2 + 4(t-2) + 4]\, u(t-2)\}$

$= \mathcal{L}\{(t-\cancel{2})^2 u(t-\cancel{2}) + 4(t-\cancel{2})u(t-\cancel{2}) + 4\, u(t-\cancel{2})\}$

$= \dfrac{2!}{s^3} e^{-2s} + 4\, \dfrac{1}{s^2} e^{-2s} + \dfrac{4}{s} e^{-2s}$

Ex:- $\mathcal{L}\{(3t+1)u(t-3)\}$

Soln

$3t = 3[(t-3)+3] = 3(t-3) + 9$

$\mathcal{L}\{(3t+1)u(t-3)\} = \mathcal{L}\{[3(t-3)+9+1]\, u(t-3)\}$

$= \mathcal{L}\{[3(t-3)+10]\, u(t-3)\}$

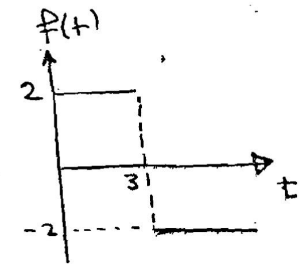
$\mathcal{L}\{3(t-\cancel{3})u(t-\cancel{3})\} + \mathcal{L}\{10\, u(t-\cancel{3})\}$

$\dfrac{3}{s^2} e^{-3s} + \dfrac{10}{s} e^{-3s}$

Important examples

Example (1) Express $f(t)$ in terms of a unit step, the[n] Find its L.T

$$f(t) = \begin{cases} 2 & 0 \leq t \leq 3 \\ -2 & t > 3 \end{cases}$$



Soln:- $f(t) = 2 + (-2-2)\,u(t-3)$

$$f(t) = 2 - 4u(t-3)$$

$$\mathcal{L}\{2 - 4u(t-3)\} = \frac{2}{s} - \frac{4}{s} e^{-3s}$$

Ex(2) $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4 & t > 2 \end{cases}$

Soln $f(t) = t^2 + (4 - t^2)\,u(t-2)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 4u(t-2) - t^2 u(t-2)\}$$

$$t^2 = [(t-2) + 2]^2 = (t-2)^2 + 4(t-2) + 4$$

$$= \mathcal{L}\{t^2 + \cancel{4u(t-2)} - (t-2)^2 u(t-2) - 4(t-2)u(t-2) \cancel{- 4u(t-2)}\}$$

$$\frac{2!}{s^3} - \frac{2!}{s^3} e^{-2s} - \frac{4}{s^2} e^{-2s}$$

Ex(3) $f(t) = \begin{cases} e^{-t} & 0<t<2 \\ 0 & t>2 \end{cases}$

Soln $f(t) = e^{-t} + (0 - e^{-t})u(t-2)$

$$f(t) = e^{-t} - e^{-t}u(t-2)$$

$$t = (t-2)+2$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t} - e^{-[(t-2)+2]}u(t-2)\}$$

ثابت

$$= \mathcal{L}\{e^{-t} - e^{-(t-\cancel{2})}e^{-2}u(t-\cancel{2})\}$$

$$= \frac{1}{s+1} - e^{-2}\frac{1}{s+1}e^{-2s}$$

---

xample $\mathcal{L}\{e^{2t}u(t-3)\}$

ln $t = (t-3)+3$

$$^{2t}u(t-3)\} = \mathcal{L}\{e^{2[(t-3)+3]}u(t-3)\}$$

ث

$$= \mathcal{L}\{e^{6}e^{2(t-\cancel{3})}u(t-\cancel{3})\}$$

$$e^{6}\frac{1}{s-2}e^{-3s}$$

Example: Find L.T for $f(t) = e^{t} \sin t \; u(t-\pi)$

Soln.

$$t = (t-\pi)+\pi$$

$$\sin t = \sin[(t-\pi)+\pi]$$

$$= \sin(t-\pi)\underbrace{\cos\pi}_{-1} + \underbrace{\sin\pi}_{0}\cos(t-\pi)$$

$$= -\sin(t-\pi)$$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\left\{ e^{(t-\pi)+\pi}\left(-\sin(t-\pi)\right) u(t-\pi)\right\}$$

$$= \mathcal{L}\left\{ -e^{\pi}\, e^{t\cancel{-\pi}} \sin(t\cancel{-\pi})\, u(t\cancel{-\pi})\right\}$$

$$= -e^{\pi}\, \mathcal{L}\{e^{t}\sin t\}\,.\, e^{-\pi s}$$

$$= -e^{\pi} \; \frac{1}{(s-1)^2+1} \; e^{-\pi s}$$