2

Laplace Transform

* Recall

In case of $f(t) = t^n$

اذا كانت n ← عدد صحيح موجب

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

* (اذا كانت n كسر)

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$\Gamma(n+1) \longrightarrow$ is called Gamma function

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(n+1) = n! \longrightarrow$ n +ve integer

$\Gamma(1) = 1 \quad , \quad \Gamma(1/2) = \sqrt{\pi} \Rightarrow$ حفظ

Ex find $\mathcal{L}\{t^{3/2}\}$

Soln $\mathcal{L}\{t^{3/2}\} = \dfrac{\Gamma(5/2)}{s^{5/2}}$

$\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\,\frac{1}{2}\,\Gamma(\frac{1}{2})$

$= \frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}$

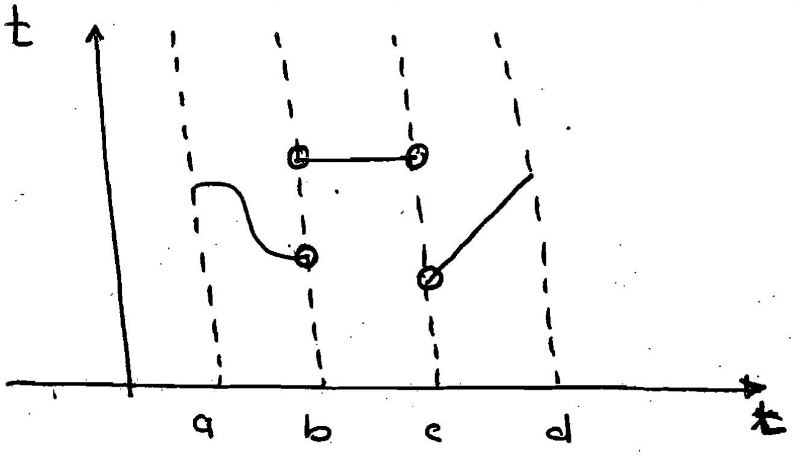
$\therefore \mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}/4}{s^{5/2}}$

Ex $\mathcal{L}\{\frac{1}{\sqrt{t}}\}$

Soln $\mathcal{L}\{t^{-1/2}\} = \dfrac{\Gamma(1/2)}{s^{1/2}}$

$= \dfrac{\sqrt{\pi}}{s^{1/2}}$

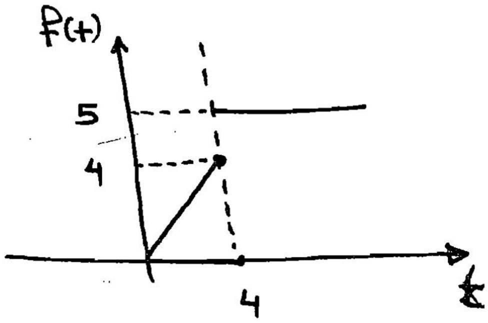
* Piecewise Continous function



$a \leq t \leq d$

Example :- Find laplace Transform for the Function

$$f(t) = \begin{cases} t & 0 \leq t < 4 \\ 5 & t \geq 4 \end{cases}$$



Soln

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^4 e^{-st}\, t\, dt + \int_4^\infty e^{-st}(5)\, dt$$

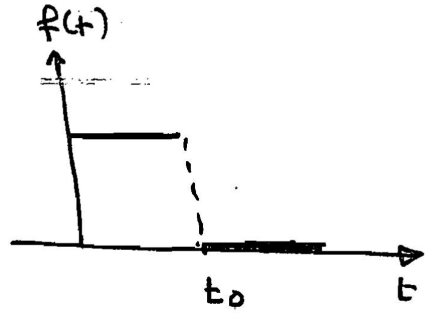
d	∫
t	e^{-st}
1	$-\frac{e^{-st}}{s}$
0	$\frac{e^{-st}}{s^2}$

$$= \left[-\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^4 + 5\left[-\frac{e^{-st}}{s}\right]_4^\infty$$

$$= \left[-\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} - \left(0 - \frac{1}{s^2}\right)\right] + 5\left[0 + \frac{e^{-4s}}{s}\right]$$

$$F(s) = \frac{-4}{s} e^{-4s} - \frac{1}{s^2} e^{-4s} + \frac{1}{s^2} + \frac{5}{s} e^{-4s}$$

Ex:- Find L.T For $f(t) = 2 \quad 0 \leq t < t_0$
$\qquad\qquad\qquad\qquad\quad = 0 \quad t \geq t_0$

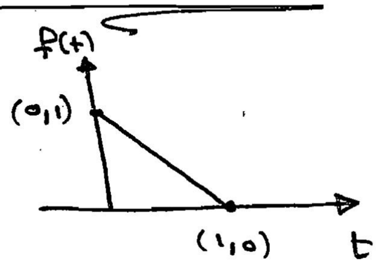


Soln $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^{t_0} e^{-st}(2)\, dt + \int_{t_0}^{\infty} e^{-st}(0)\, dt \xrightarrow{} 0$$

$$= \frac{-2}{s}\left[e^{-st}\right]_0^{t_0} = \frac{-2}{s}\left(e^{-st_0} - 1\right)$$

Ex:- Find L.T for



Soln First we must Find the equation of straight line معادلة خط مستقيم

$$\Rightarrow \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \rightsquigarrow \text{معادلة لنقطتين } (x_1, y_1), (x_2, y_2)$$

$$\frac{y - 1}{x - 0} = \frac{0 - 1}{1 - 0} \qquad y - 1 = -x \qquad \boxed{y = 1 - x}$$

$$\therefore f(t) = \begin{cases} 1 - t & 0 \leq t < 1 \\ 0 & t > 1 \end{cases}$$

Some Important Rules

* $\ln A + \ln B = \ln AB$
* $\ln A - \ln B = \ln A/B$
* $n \ln A = \ln A^n$
* $-\ln A = \ln \frac{1}{A}$
* $\ln 1 = 0$

* $\tan^{-1} \infty = \pi/2$
* $\tan^{-1} 0 = 0$
* $\tan^{-1} 1 = \pi/4$
* $\frac{1}{\infty} = 0$
* $\cot^{-1} x = \tan^{-1} \frac{1}{x} = \pi/2 - \tan^{-1} x$

Note مهم

* $$\int_4^\infty \left(\frac{1}{x-3} - \frac{1}{x+1} \right) dx$$

$$= \left[\ln(x-3) - \ln(x+1) \right]_4^\infty = \left[\ln \frac{x-3}{x+1} \right]_4^\infty \div \frac{x}{x}$$

$$= \left[\underbrace{\ln \frac{\infty}{\infty}}_{\text{كمية غير معرفة}} - \ln \frac{1}{5} \right]$$

في هذه الحالة (كمية غير معرفة) نقسم البسط والمقام على أعلى أس (power) لـ (x)

$$I = \left[\ln \frac{1 - 3/x}{1 + 1/x} \right]_4^\infty = \left[\underbrace{\ln 1}_{0} - \ln 1/5 \right] = -\ln 1/5$$

* Properties of Laplace Transform

II 1 Derivative of The transform (Multiply by t)

If $\mathcal{L}\{f(t)\} = F(s)$

prove That $\mathcal{L}\{t f(t)\} = (-1)\dfrac{dF(s)}{ds}$

Soln:- $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-t)\, e^{-st} f(t)\, dt$$

$$\frac{dF(s)}{ds} = -\int_0^\infty e^{-st}\, t f(t)\, dt$$

$$\therefore \frac{dF(s)}{ds} = -\mathcal{L}\{t f(t)\}$$

$$\therefore \boxed{\begin{aligned} \mathcal{L}\{t f(t)\} &= -\frac{dF(s)}{ds} \\ \mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n F(s)}{ds^n} \end{aligned}}$$

Ex: Find L.T for $F(t) = t\cos 3t$

Soln

$$\mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}F(s)$$

$$\cos 3t \xrightarrow{\mathcal{L}} \frac{s}{s^2+9}$$

$$\therefore \mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}\frac{s}{s^2+9}$$

Note $$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\therefore \mathcal{L}\{t\cos 3t\} = -\frac{(s^2+9) - s(2s)}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}$$

Note

ثابت

$$\frac{A}{u} \xrightarrow[\text{تفاضل}]{d/dx} \frac{-A}{u^2}\,u'$$

$$\frac{u}{v} \xrightarrow[\text{تفاضل}]{d/dx} \frac{vu' - v'u}{v^2}$$

Ex Find $\mathcal{L}.T$ for $f(t) = t\sin^2 2t$

Soln

$$\mathcal{L}\{t \sin^2 2t\} = -\frac{d}{ds} F(s)$$

$$\sin^2 2t = \frac{1}{2}(1-\cos 4t) \xrightarrow{\mathcal{L}.T} \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$\mathcal{L}\{t\sin^2 2t\} = -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right)$$

Ex Find $\mathcal{L}.T$ for $f(t) = t e^{2t}\cosh t$

Soln $f(t) = t\, e^{2t} \frac{1}{2}(e^{t}+e^{-t}) = \frac{1}{2} t (e^{3t}+e^{t})$

$$\frac{1}{2}(e^{3t}+e^{t}) \xrightarrow{\mathcal{L}.T} \frac{1}{2}\left(\frac{1}{s-3} + \frac{1}{s-1}\right)$$

$$\frac{1}{2}t(e^{3t}+e^{t}) \xrightarrow{\mathcal{L}.T} -\frac{1}{2}\left(\frac{-1}{(s-3)^2} \mp \frac{1}{(s-1)^2}\right)$$

$$\therefore \mathcal{L}\{te^{2t}\cosh t\} = -\frac{1}{2}\left(-\frac{1}{(s-3)^2} \mp \frac{1}{(s-1)^2}\right)$$

Examples

① $\mathcal{L}\{e^{t}\cosh^{2}2t\}$

Soln $\mathcal{L}\left\{e^{t}\left(\dfrac{e^{2t}+e^{-2t}}{2}\right)^{2}\right\}$

$$=\frac{1}{4}\mathcal{L}\{e^{t}\cdot(e^{4t}+e^{-4t}+2)\}$$

$$=\frac{1}{4}\mathcal{L}\{e^{5t}+e^{-3t}+2e^{t}\}$$

$$=\frac{1}{4}\left[\frac{1}{s-5}+\frac{1}{s+3}+2\,\frac{1}{s-1}\right]$$

② $\mathcal{L}\{t^{2}\sin 2t\}$

① $\mathcal{L}\{\sin 2t\}=\dfrac{2}{s^{2}+4}$

② $\mathcal{L}\{t^{2}\sin 2t\}=(-1)^{2}\dfrac{d^{2}}{ds^{2}}\left(\dfrac{2}{s^{2}+4}\right)$

2

Integration of transform (Division by t)

If $\mathcal{L}\{f(t)\} = F(s)$

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds}$$

Take Care

* $\tan^{-1}\infty = \frac{\pi}{2}$

* $\int \frac{1}{s^2+a^2}\,ds = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$

* $\int \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\ln(s^2+a^2) + c$

Note That

$$\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_{s}^{\infty} F(s)\, ds$$

Under Condition

$$\lim_{t \to 0} \frac{f(t)}{t} = \text{Exist}$$

Ex $f(t) = \frac{\sin t}{t}$ $\lim_{t \to 0} \frac{\sin t}{t} = 1$

Exist

Ex $f(t) = \frac{\cos t}{t}$ $\lim_{t \to 0} \frac{\cos t}{t} = \infty$

doesn't exist

Ex Find $\mathcal{L}.T$ for $f(t) = \dfrac{e^{3t} - e^{-t}}{t}$

Soln check $\lim\limits_{t \to 0} \dfrac{e^{3t} - e^{-t}}{t} = \dfrac{0}{0}$

$$\lim_{t \to 0} \frac{3e^{3t} + e^{-t}}{1} = 4 \to \text{Exist}$$

$$\therefore \mathcal{L}\left\{ \frac{e^{3t} - e^{-t}}{t} \right\} = \int_{s}^{\infty} F(s)\, ds$$

$$\mathcal{L}\{ e^{3t} - e^{-t} \} = \frac{1}{s-3} - \frac{1}{s+1}$$

$$\mathcal{L}\left\{ \frac{e^{3t} - e^{-t}}{t} \right\} = \int_{s}^{\infty} \frac{1}{s-3} - \frac{1}{s+1}\, ds$$

$$= \Big[\ln(s-3) - \ln(s+1) \Big]_{s}^{\infty} = \left[\ln \frac{s-3}{s+1} \right]_{s}^{\infty}$$

$$= \left[\ln \frac{1 - 3/s}{1 + 1/s} \right]_{s}^{\infty} = \left[\cancel{\ln 1}_{\to 0} - \ln \frac{s-3}{s+1} \right]$$

$$= \ln \frac{s+1}{s-3} \quad \#$$

$*\ F(t) = \dfrac{1 - \cos 2t}{t}$

Check: $\lim\limits_{t \to 0} \dfrac{1-\cos 2t}{t} = \dfrac{0}{0}$

$= \lim\limits_{t \to 0} \dfrac{+2\sin 2t}{1} = 0$ Exist

Soln

$$\mathcal{L}\left\{\frac{1 - \cos 2t}{t}\right\} = \int_{s}^{\infty} F(s)\, ds'$$

$$1 - \cos 2t \xrightarrow{\mathcal{L}} \frac{1}{s} - \frac{s}{s^2+4}$$

$$\mathcal{L}\left\{\frac{1 - \cos 2t}{t}\right\} = \int_{s'}^{\infty}\left(\frac{1}{s} - \frac{s}{s'^2+4}\right) ds'$$

$$= \left[\ln s - \frac{1}{2}\ln(s'^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[2\ln s - \ln(s^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[\ln s^2 - \ln(s'^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[\ln \frac{s^2}{s'^2+4}\right]_{s'}^{\infty} =$$

بالقسمة على اعلى اس ل (s) وهو s^2

$$= \frac{1}{2}\left[\ln \frac{1}{1+4/s^2}\right]_{s}^{\infty} = \frac{1}{2}\left[\ln 1 - \ln \frac{s'^2}{s^2+4}\right]$$

$$= -\frac{1}{2}\ln \frac{s'^2}{s^2+4} = \frac{1}{2}\ln \frac{s^2+4}{s^2}$$