Chapter five

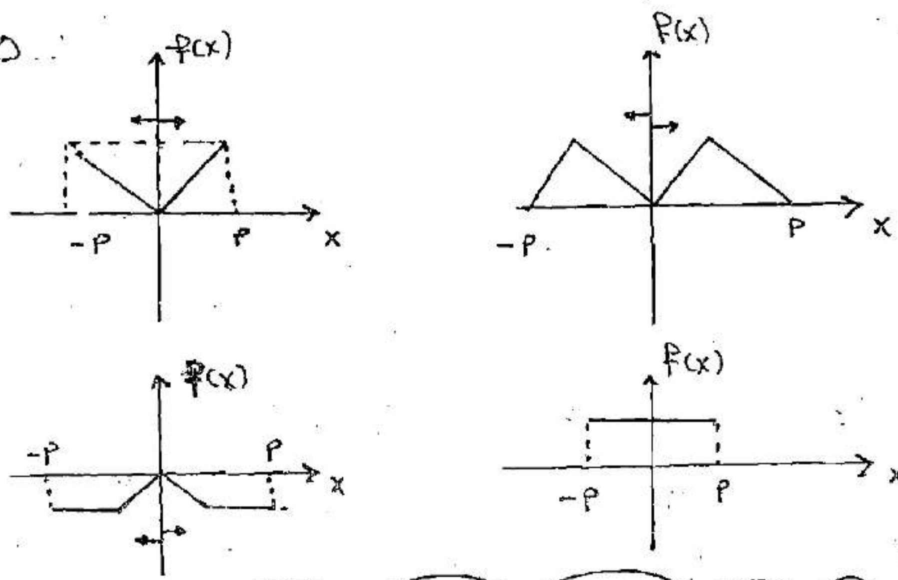
Fourier Series

Some Important Notes

Even function - $f(x) = f(-x)$ الداله الزوجيه

الداله الزوجيه تكون متماثله حول محور (y)

Examples

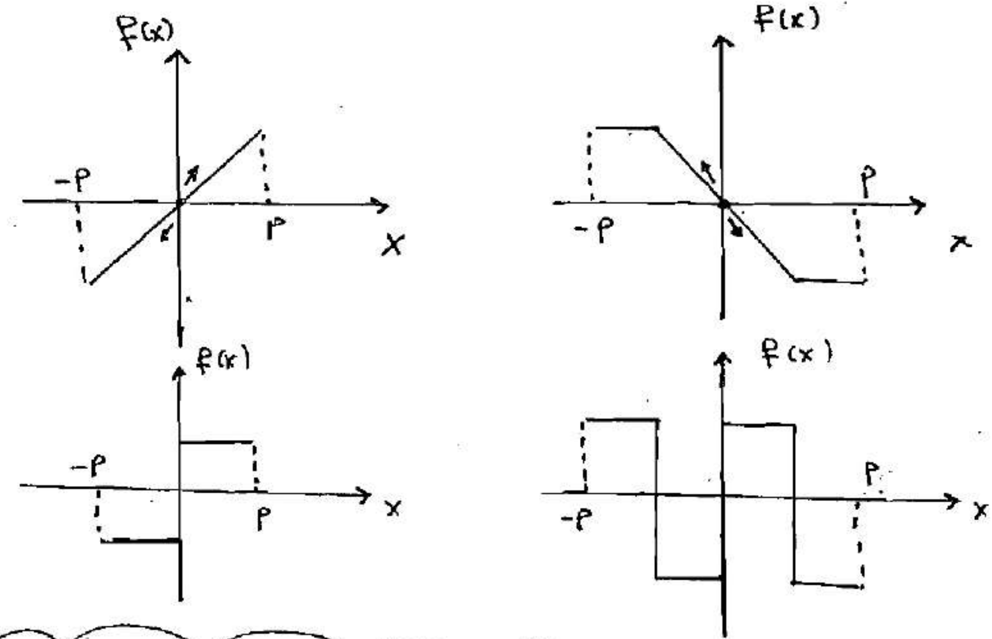


$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx$$

Odd function :- $f(-x) = -f(x)$ الدالة الفردية

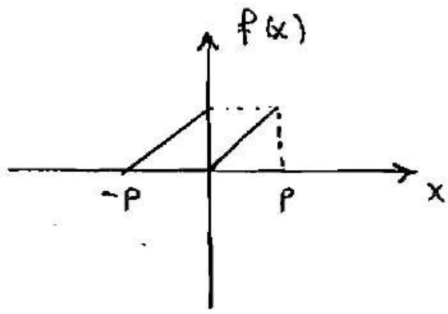
الدالة الفردية تكون متماثلة حول نقطة الأصل (0)

Examples



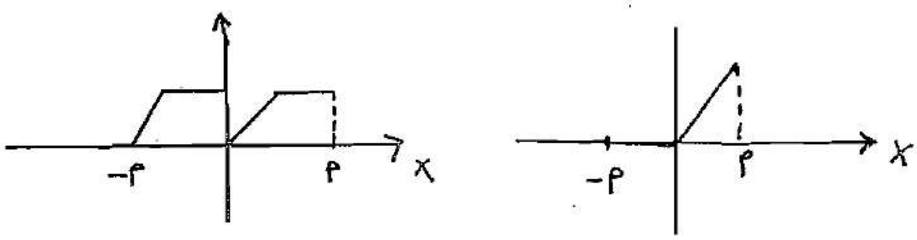
$$\int_{-p}^{p} f(x)\,dx = 0$$

Not odd / Not even



ليست متماثلة حول محور (y)

وكذلك ليست متماثلة حول نقطة الأصل



Note That

$$\sin n\pi = 0$$

جيد

$$n = 0, \pm 1, \pm 2, \cdots$$

$$\cos n\pi = (-1)^n \quad \text{for all } n$$

$$\cos n\pi = \begin{cases} 1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases}$$

even

$$\cos 2n\pi = 1$$

Periodic Function :- $f(x) = f(x+\omega)$

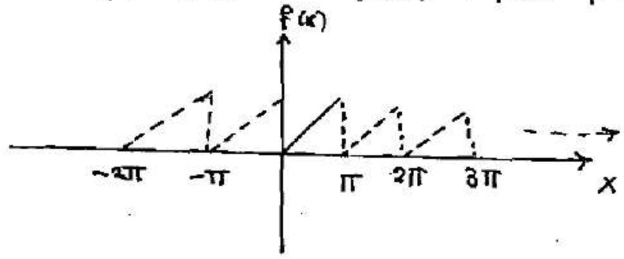
الداله التكراريه (تتكرر كل (ω))

اى ω هى الفتره المكامله (الدوره كامله)

ω = full period

Examples

① $f(x) = x \quad 0 < x < \pi \qquad f(x) = f(x+\pi)$

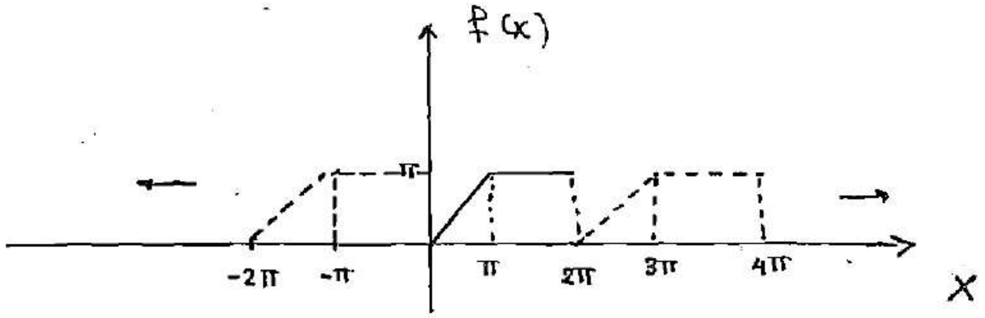


نلاحظ انها ليست زوجيه وليست فرديه

انقل محور (y) عند نهاية الفتره , اعتبره كأنه نقطة الاصل , وارسم الداله مره اخرى

② $f(x) = x \quad 0 < x < \pi \qquad f(x) = f(x+2\pi)$

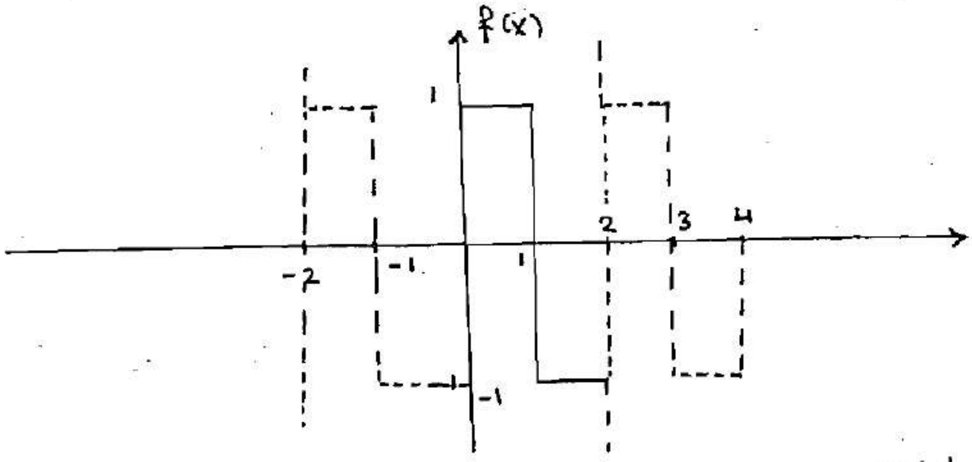
$\quad\;\; = \pi \quad \pi \leq x < 2\pi$



not odd
not even

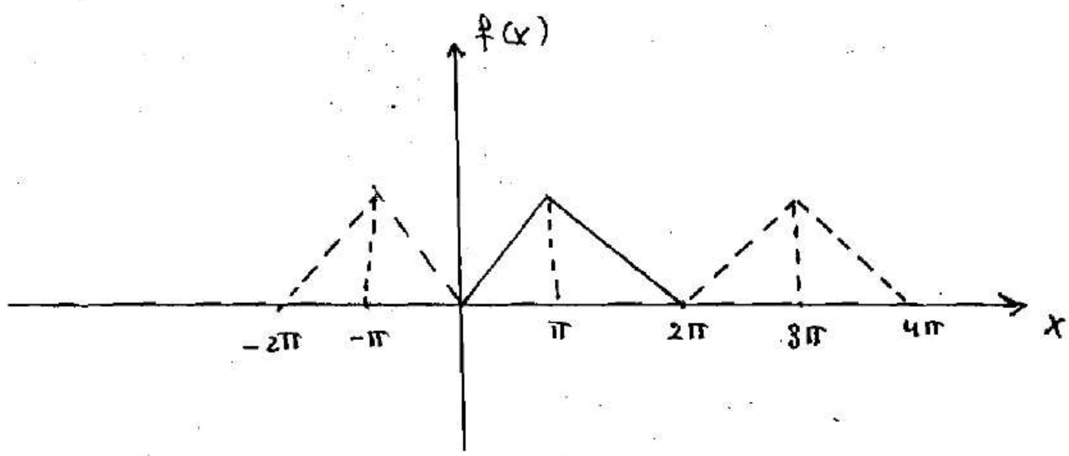
نلاحظ انها ليست فرديه وليست زوجيه

③ $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ -1 & 1 \le x < 2 \end{cases}$ $\quad f(x) = f(x+2)$



odd f_n

④ $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$ $\quad f(x) = f(x+2\pi)$



Even f_n

Fourier Expansion

If $f(x) = f(x+2p)$ periodic function of period $2p$

$2p =$ الدوره الكامله
$=$ full period

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos \frac{n\pi}{p}x + b_n \sin \frac{n\pi}{p}x\right]$$

Average area in one period

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$ [area under the Curve in one period

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos \frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin \frac{n\pi}{p}x\,dx$$

$f(x)$ odd fn	$f(x)$ Even fn	Not odd / Not even
$\boxed{\text{Sines only}}$	$\boxed{\text{Cosines only}}$	
$a_0 = 0$	$b_n = 0$	$a_0 = \frac{1}{p}\int$
$a_n = 0$	$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$	$a_n = \frac{1}{p}\int$
$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$	$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$	$b_n = \frac{1}{p}\int$
او التكامل على نصف فتره	او التكامل على نصف فتره	التكامل على الفتره المعطاه في المسأله (الفتره الكامله)

Steps of Solution

① ارسم الدالة (الرسم مهم جداً)

② حدد الفترة بالكامل $2p$

③ حدد نوع الدالة زوجية او فردية
او لا زوجية / لا فردية

④ حدد a_0, a_n, b_n

⑤ نعوض في القانون الاصلي

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$$

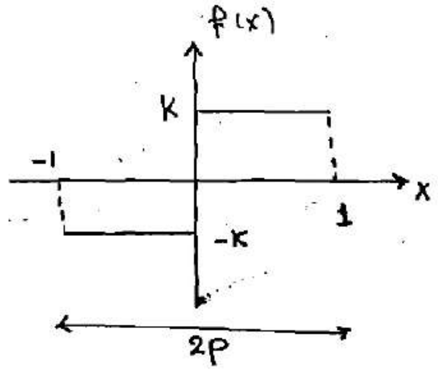
$f(x)$ odd	$f(x)$ even	neither even nor odd
$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$
Sines only	Cosines only	

Examples

① Expand in Fourier Series the f_n

$$f(x) = \begin{cases} -k & -1 < x < 0 \\ k & 0 \leq x < 1 \end{cases} \qquad f(x) = f(x+2)$$

Solution



$$2p = 2$$

$$\boxed{p = 1}$$

لاحظ ان الدالة متماثلة حول نقطة الاصل (o)

$\therefore f(x)$ is odd f_n

$\therefore a_0 = a_n = 0$ $\qquad \therefore f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx$$

$$b_n = 2 \int_0^1 K \sin n\pi x \, dx$$

$$= -2K \frac{1}{n\pi} \left(\cos n\pi x \right)_0^1$$

$$b_n = \frac{-2K}{n\pi} \left(\cos n\pi - 1 \right) = \frac{-2K}{n\pi} \left((-1)^n - 1 \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{P} x \qquad \boxed{P=1}$$

$$= \sum_{n=1}^{\infty} \frac{-2K}{n\pi}\left((-1)^n - 1\right) \sin n\pi x$$

$$f(x) = \frac{-2K}{\pi}\left(-2\sin \pi x + 0 - \frac{2}{3}\sin 3\pi x + \cdots\cdots\right)$$

OR We Can Say $b_n = \frac{-2K}{n\pi}\left((-1)^n - 1\right)$

$$b_n = \begin{cases} \frac{4K}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\therefore f(x) = \sum_{n=\text{odd}} \frac{4K}{n\pi} \sin n\pi x$$

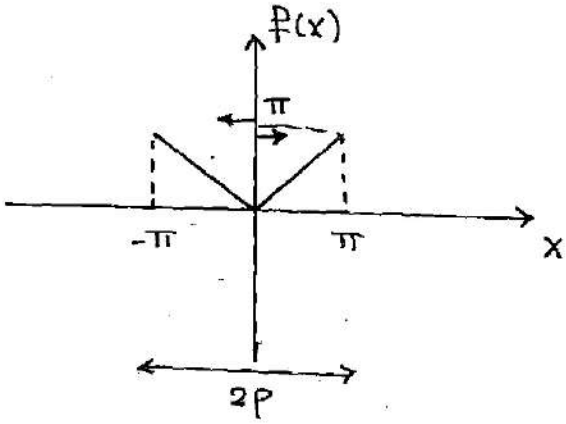
$$f(x) = \frac{4K}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x \cdots\right)$$

② Expand in Fourier series $f(x) = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x \leq 0 \end{cases}$

Soln

$2p = 2\pi$

$\boxed{p = \pi}$



$f(x)$ is even $f_{\underline{n}}$

$\boxed{b_n = 0}$ $\quad a_0 = \checkmark \quad a_n = \checkmark$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$

* $\underline{a_0}$ $\quad a_0 = \frac{2}{p} \int_0^p f(x)\, dx$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{\cancel{2}}{\pi} \left(\frac{x^2}{\cancel{2}} \right)_0^{\pi}$$

$$a_0 = \frac{\pi^{\cancel{2}}}{\cancel{\pi}} \qquad \boxed{a_0 = \pi}$$

* $\underline{a_n}$ $\quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x \, dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

d		∫
x		$\cos nx$
	$+$	
1		$\frac{1}{n}\sin nx$
	$-$	
0		$\frac{-1}{n^2}\cos nx$

$$a_n = \frac{2}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi}{n}\underbrace{\sin n\pi}_{0} + \frac{1}{n^2}\underbrace{\cos n\pi}_{(-1)^n}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$a_n = \frac{2}{\pi}\left(\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right)$$

$$a_n = \frac{2}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad \boxed{p=\pi}$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right)\cos nx$$

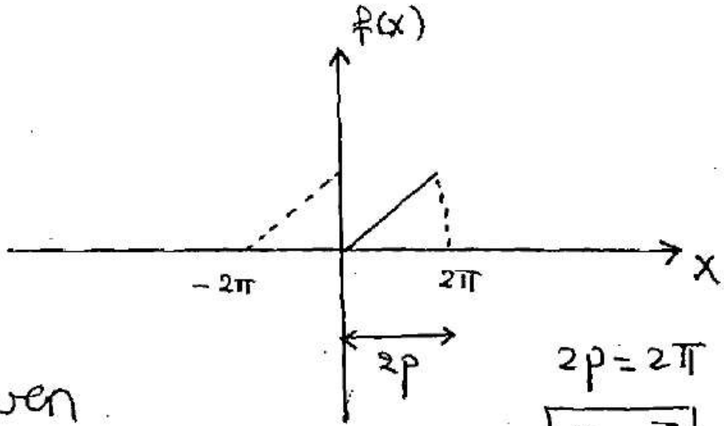
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left(-2\cos x + 0 - \frac{2}{9}\cos 3x \cdots\right)$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x \cdots\right)$$

(5) Find Fourier Series for

$$f(x) = x \qquad 0 \leq x \leq 2\pi$$

Solution



$2p = 2\pi$

$\boxed{p = \pi}$

$f(x)$ not odd / not even

$a_0 = \checkmark,\ a_n = \checkmark,\ b_n = \checkmark$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

* a_0

$$a_0 = \frac{1}{p}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}$$

$$= \frac{1}{\cancel{2\pi}}\,\cancel{4}^{2}\pi^{\cancel{2}} \qquad \boxed{a_0 = 2\pi}$$

* $\underline{a_n}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$\begin{array}{ccc} d & & \int \\ x & & \cos nx \\ 1 & \searrow^{+} & \frac{1}{n}\sin nx \\ 0 & \searrow^{-} & -\frac{1}{n^2}\cos nx \end{array}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{1}{n}2\pi\underbrace{\sin 2n\pi}_{0} + \frac{1}{n^2}\underbrace{\cos 2n\pi}_{1}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$a_n = \frac{1}{\pi}\underbrace{\left(\frac{1}{n^2} - \frac{1}{n^2}\right)}_{0}$$

$$\boxed{a_n = 0}$$

* $\underline{b_n}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$\begin{array}{ccc} x & & \sin nx \\ 1 & \searrow^{+} & -\frac{1}{n}\cos nx \\ 0 & \searrow^{-} & -\frac{1}{n^2}\sin nx \end{array}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

$$b_n = \frac{1}{\cancel{\pi}}\left[\left(-\frac{1}{n}2\cancel{\pi}\underbrace{\cos 2n\pi}_{1} + \frac{1}{n^2}\underbrace{\sin 2n\pi}_{0}\right) - 0\right]$$

$$b_n = -\frac{2}{n}$$

$$f(x) = \frac{2\pi}{2} + \sum \cancel{0} + \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$$

$\to 0$

$$f(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \cdots\right)$$