Non Homogenous 2^{nd} Order D.E (const. Coeff)

Consider the non-homogeous equ. as

$$a.y'' + b.y' + c.y = F(x)$$

D-operator

Let $\frac{d}{dx} = D$ $\therefore y' = \frac{dy}{dx} = D.y$

$\frac{d^2}{dx^2} = D^2$ $\therefore y'' = \frac{d^2y}{dx^2} = D^2.y$

The equ. becomes

$$\underbrace{(a.D^2 + b.D + c).y = F(x)}$$

The Solution given by :

$$\boxed{y_{G.S.} = y_{C.F.} + y_{P.I.}}$$

where:

$y_{G.S.}$ = General Solution of D.E.

$y_{C.F.}$ = Complementary function Solution

y = Particular Integral Solution

To get $y_{c.f.}$

$y_{c.f.} \rightarrow$ The Solution of the homogeneous D.E.

To get $y_{p.I.}$

$$\because \ (\underbrace{aD^2 + b.D + c}_{P(D)}) \cdot y = F(x)$$

$P(D) \longrightarrow$ diff. polynomial

$$\therefore \ P(D) \cdot y = F(x)$$

$$\therefore \ y_{p.I.} = \frac{1}{P(D)} \cdot F(x)$$

Properties :

① $y = \frac{1}{P(D)} \cdot [c\, F(x)] = c \cdot \frac{1}{P(D)} \cdot F(x)$

② $y = \frac{1}{P(D)} \cdot [F_1(x) + F_2(x)] = \frac{1}{P(D)} \cdot F_1(x) + \frac{1}{P(D)} \cdot F_2(x)$

③ $y = \frac{1}{P_1(D) \cdot P_2(D)} \cdot [F(x)] = \frac{1}{P_1(D)} \cdot \left[\frac{1}{P_2(D)} \cdot F(x) \right]$

1] if $f(x) = e^{mx}$

$$\therefore y_{P.I.} = \frac{1}{P(D)} \cdot e^{mx}$$

Put $D \longrightarrow m$

Prove)

$$\because D \cdot e^{mx} = m \cdot e^{mx}$$

$$\because D^2 \cdot e^{mx} = m^2 \cdot e^{mx}$$

$$\because D^3 \cdot e^{mx} = m^3 \cdot e^{mx}$$

$$\therefore D^n \cdot e^{mx} = m^n \cdot e^{mx}$$

Special Case

if $f(x) = \text{constant} = c \cdot e^{0 \cdot x}$

$\therefore D \longrightarrow 0$

Example : ① Find $y_{P.I.}$ for the DE.

$$(D^2 - 4D + 4) \cdot y = 2 \cdot e^{3x}$$

Solution

$$\because y_{P.I.} = \frac{1}{D^2 - 4D + 4} \cdot 2e^{3x}$$

$$\therefore \boxed{D \longrightarrow 3}$$

$$\therefore y_{P.I.} = 2 \cdot \frac{1}{9 - 12 + 4} \cdot e^{3x} =$$

$$\therefore y_{P.I.} = 2e^{3x}$$

Example : ② Find $y_{P.I.}$ for the DE

$$(D^2 - 2D + 2) \cdot y = 8$$

Solution

$$\because y_{P.I.} = \frac{1}{\cancel{D^2} - 2\cancel{D} + 2} \cdot 8 \qquad \boxed{\therefore D \rightarrow 0}$$

$$\therefore y_{P.I.} = 4$$

Example : 3 Solve for $y_{P.I.}$

$$(D^2 + 2D) \cdot y = 6$$

Solution

$$\therefore \; y_{P.I.} = \frac{1}{D^2 + 2D} \cdot (6)$$

constant

$D \to 0$

$$\therefore \; y_{P.I.} = \frac{1}{0} \cdot (6) = \alpha \; \alpha \; \alpha$$

$$\boxed{D = \frac{d}{dx}}$$

$$\therefore \; y_{P.I.} = \frac{1}{D} \cdot \left[\frac{1}{\cancel{D} + 2} (6) \right]$$

$D \to 0$

$$\boxed{\frac{1}{D} = \int dx}$$

$$\therefore \; y_{P.I.} = \frac{1}{D} \cdot (3)$$

$$\therefore \; y_{P.I.} = \int 3 \cdot dx = 3 \cdot x$$

2] if $F(x) = \sin(mx)$ or $\cos(mx)$

$$\because y_{P.I.} = \frac{1}{P(D)} \cdot \begin{pmatrix} \sin(mx) \\ \cos(mx) \end{pmatrix}$$

Put $D^2 \longrightarrow -m^2$

Prove)

$$\because D \cdot \sin(mx) = m \cdot \cos(mx)$$

$$\because D^2 \cdot \sin(mx) = -m^2 \cdot \sin(mx)$$

$$D^2 \longrightarrow -m^2$$

$$\because D^3 \cdot \sin(mx) = -m^3 \cdot \cos(mx)$$

$$\because D^4 \cdot \sin(mx) = m^4 \cdot \sin(mx)$$

$$D^4 \longrightarrow m^4$$

Example : ③ Find $y_{P.I.}$ for the D.E.

$$y'' + 9 \cdot y = \sin(2x)$$

Solution

$$\because (D^2 + 9) \cdot y = \sin(2x)$$

$$\therefore y_{P.I.} = \frac{1}{D^2 + 9} \cdot (\sin 2x) \qquad D^2 \to -(2)^2$$

$$\therefore D^2 \to -4$$

$$\therefore y_{P.I.} = \frac{1}{-4+9} \cdot \sin 2x = \frac{1}{5} \cdot \sin 2x$$

Example : ④ Find $y_{P.I.}$ for

$$(D^2 + 4) y = 10 \cdot \cos(x)$$

Solution

$$\because y_{PI} = \frac{1}{\cancel{D^2} + 4} \cdot 10 \cos(x)$$

$$-1$$

$$\therefore D^2 \to -(1)^2$$

$$\therefore D^2 \to -1$$

$$\therefore y_{P.I.} = \frac{10}{3} \cdot \cos x$$

Example : ⑤ Find $y_{P.I.}$ for the D.E.

$$(D^2+1)\cdot y = \sinh(2x)$$

Solution

$$\because y_{P.I.} = \frac{1}{D^2+1} \cdot \sinh(2x)$$

$$D^2 \to (2)^2$$

$$\therefore D^2 \to 4$$

$$\therefore y_{P.I.} = \frac{1}{4+1} \cdot \sinh(2x) = \frac{1}{5} \cdot \sinh(2x)$$

Example : ⑥ find $y_{P.I.}$ for the D.E.

$$(D^2-4)\cdot y = 2\cdot\cosh(4x)$$

Solution

$$\because y_{P.I.} = \frac{1}{D^2-4} \cdot 2\cosh(4x)$$

$$D^2 \to (4)^2$$

$$\therefore D^2 \to 16$$

$$\therefore y_{P.I.} = \frac{1}{16-4} \cdot 2\cosh(4x)$$

$$\therefore y_{P.I.} = \frac{1}{6} \cdot \cosh(4x)$$

4) if $F(x)$ = Polynomial function

$$\therefore F(x) = a_0 \cdot x^{n} + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \cdots + a_n$$

كثيرة حدود من الدرجة (n)

$$\because y_{P.I.} = \frac{1}{P(D)} \cdot \left[a_0 \cdot x^{n} + a_1 \cdot x^{n-1} + \cdots + a_n \right]$$

Using binomial Theorem

Using Long division

Recall

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 + \cdots$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2 \cdot x + 3 \cdot x^2 - 4 \cdot x^3 + \cdots$$

Example : 7 Find $y_{P.I.}$ for the DE.

$$(D^2 - 2D + 1)\cdot y = x^2 + 3x + 4$$

Solution

$$\because\ y_{P.I.} = \frac{1}{D^2 - 2D + 1}\cdot\left(x^2 + 3x + 4\right)$$

كثيرة حدود من الدرجة (2)

using Long division

Stop

لازم ترتيبها

$$\begin{array}{r|l}
 & 1 + 2D + 3D^2 \\
\hline
1 - 2D + D^2 & 1 \\
 & \cancel{-1} \pm 2D \mp D^2 \\
\hline
 & 2D - D^2 \\
 & \cancel{-2D} \pm 4D^2 \mp 2D^3 \\
\hline
 & 3D^2 - 2D^3
\end{array}$$

$\div$

$\div$

$$\therefore\ y_{P.I.} = \left(1 + 2D + 3D^2 + \cdots\right)\cdot\left(x^2 + 3x + 4\right)$$

$$\therefore\ y_{P.I.} = 1\cdot\left(x^2 + 3x + 4\right) + 2\cdot\left(2x + 3\right) + 3\cdot(2)$$

Example : (8) Find $y_{P.I.}$ for the DE.

$$(D^2+1)\cdot y = x^4 + 2$$

Solution

$$\because \; y_{P.I.} = \frac{1}{D^2+1}\,(x^4+2)$$

كثرة حدود من الدرجة (4)

long division   Binomial Theorem

$$\because \; \frac{1}{D^2+1} = (1+D^2)^{-1} = 1 - D^2 + D^4 - D^6 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\therefore \; y_{P.I.} = (1 - D^2 + D^4 - \cdots)\cdot(x^4+2)$$

$$\therefore \; y_{P.I.} = 1\cdot(x^4+2) - (12\cdot x^2) + (24)$$

$$\therefore \; y_{P.I.} = x^4 - 12\cdot x^2 + 26$$

Exercise

Find $y_{P.I.}$ for the D.E.s

① $(D^2 - 2D + 3) \cdot y = 2 \cdot e^{x}$

② $(D^2 - 2D) \cdot y = e^{-4x}$

③ $y'' + 2y' = 4$

④ $y'' + 16y = 4 . \cos(3x)$

⑤ $y'' - 8 \cdot y = 2 . \cosh(2x)$

⑥ $(D^2 - 3D + 1) y = 2x^2 + 5x$

⑦ $(D^2 + 3D + 2) y = x^2 + 8$

⑧ $(D-1)^2 \cdot y = x^3 + 8$