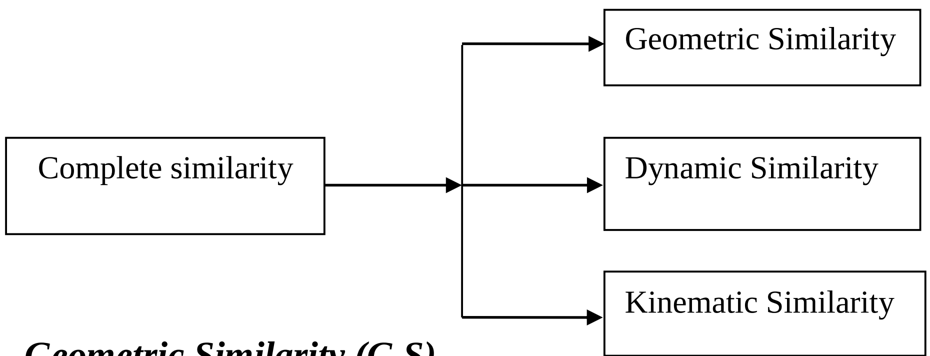
Lecture-One

Dynamic Similarity

1- *Definition of Physical Similarity*.

Two systems described by the same physics operating under different set of conditions are said to be physically similar in respect of certain specified physical quantities, when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

There are three types of similarities as in following chart which constitute the complete similarity between problems of same kind.



2- *Geometric Similarity (G.S).*

Geometric similarity implies the similarity of shape such that, the ratio of any length in one system to the corresponding length in other system is the same everywhere.

- Prototype:- is the full size or actual scale systems.
- Models:- is the laboratory scale systems.
- The model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity and properties of the fluid.

$$length\ ratio = \frac{length\ of\ model}{length\ of\ prototype}$$

Or $L_r = \frac{L_m}{L_p}$

$$area\ ratio = \frac{model\ area}{prototype\ area} = \frac{L_m^2}{L_p^2}$$

$L^2 = L_r^2$

L_r Known as the model ratio or is the scale factor

3- *Kinematic Similarity (K.S).*

Kinematic similarity refers to similarity of motion

Motion → Distance → Similarity of length (i.e., G.S.)

Motion → Time → Similarity of time intervals.

$$velocity\ ratio = \frac{V_m}{V_p} = \frac{L_m/T_m}{L_p/T_p} = \frac{L_m}{L_p} \div \frac{T_m}{T_p} = \frac{L_r}{T_r}$$

$$accelartion\ ratio = \frac{L_m/{T_m}^2}{L_p/{T_p}^2} = \frac{L_m}{L_p} \div \frac{T_m^2}{T_p^2} = \frac{L_r}{T_r^2}$$

$$flow\ rate\ ratio = \frac{Q_m}{Q_p} = \frac{{L_m}^3/T_m}{{L_p}^3/T_p} = \frac{L_m^3}{L_P^3} \div \frac{T_m}{T_p} = \frac{L_r^3}{T_r}$$

Therefore, geometric similarity is a necessary condition for the kinematic similarity to be achieved, but not the sufficient one.

4- *Dynamic Similarity (D.S).*

Dynamic similarity is the similarity of forces, in dynamically similar system, the magnitudes of forces at correspondingly similar point in each system are in a fixed ratio. In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows;

- Viscous force (due to viscosity) $\vec{F}_v$
- Pressure force (due to different in pressure) $\vec{F}_p$
- Gravity force (due to gravitational attraction) $\vec{F}_g$
- Capillary force (due to surface tension) $\vec{F}_c$
- Compressibility force (due to elasticity) $\vec{F}_e$

According to Newton's law, the resultant F_R of all these forces, will cause the accelartion of a fluid element, hence

$$\vec{F}_R = \vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e \qquad (1)$$

The inertia force $\vec{F}_i$ is defined as equal and opposite to the resultant accelerating force $\vec{F}_R$

$$\vec{F}_i = -\vec{F}_R$$

$\therefore Eq. (6.1)\ can\ be\ expressed\ as$

$$\vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e + \vec{F}_i = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both prototype and the model. The inertia force $\vec{F}_i$ is usually taken as the common one to describe the ratio

$$\frac{|\vec{F}_v|}{|\vec{F}_i|}, \frac{|\vec{F}_p|}{|\vec{F}_i|}, \frac{|\vec{F}_g|}{|\vec{F}_i|}, \frac{|\vec{F}_c|}{|\vec{F}_i|}, \frac{|\vec{F}_e|}{|\vec{F}_i|}$$

a- Inertia Force.

The inertia force is the force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.

Mass of element $\propto \rho L^3$, ρ is the density, L is the characteristic length, acceleration of a fluid element is the rate change of velocity in that direction change with time

$$a \propto \frac{V}{t}\ ; \quad t \propto \frac{L}{V}$$

$$\therefore\ a \propto \frac{V^2}{L}$$

The magnitude of inertia force is thus proportional to $\rho\, L^3 \frac{V^2}{L} = \rho\, L^2 V^2$

This can be written as $|\vec{F}_i| \propto \rho L^2 V^2$ (2)

b- Viscous Force.

The viscous force arises from shear stress in a flow of fluid, therefor, we can write magnitude of viscous force $\vec{F}_v$= shear stress * surface area.

Shear stress = μ(viscosity) * rate of shear strain

Where rate of shear strain $\propto$ velocity gradient $\propto \frac{V}{L}$

Surface area $\propto L^2$

$$\left|\vec{F}_v\right| \propto \mu \frac{V}{L} L^2 \propto \mu V L \qquad (3)$$

c- ***Pressure Force.***

The pressure force arises due to the difference of pressure in a flow field. Hence it can be written as

$$\left|\vec{F}_P\right| \propto \Delta p L^2 \qquad (4)$$

Where Δp is some characteristic pressure in the flow.

d- ***Gravity Force.***

The gravity force on a fluid element is its weight, hence,

$$\left|\vec{F}_g\right| \propto \rho L^3 g \qquad (5)$$

Where g is the acceleration due to gravity (or weight per unit mass)

e- ***Capillary or Surface Tension Force.***

The capillary force arises due to the existence of an interface between two fluids. It is equal to the coefficient of surface tension σ multiplied by the length of a linear element on the surface perpendicular to which the force acts, therefore,

$$\left|\vec{F}_c\right| \propto \sigma l \qquad (6)$$

f- ***Compressibility or Elastic Force.***

For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E ($\Delta p \propto E$), this gives rise to a force known as the elastic force .

$$\left|\vec{F}_c\right| \propto E L^2 \qquad (7)$$

Note, the flow of fluid in practice does not involve all the forces simultaneously.

5- ***D. S. of Flow Governed by Viscous, Pressure and Inertia Forces.***

The ratios of the representative magnitudes of these forces with the help of Eq's (2) to (5) as follows:

$$\frac{Viscous force}{Inertia force} = \frac{\left|\vec{F}_v\right|}{\left|\vec{F}_i\right|} \propto \frac{\mu V L}{\rho V^2 L^2} = \frac{\mu}{\rho V L} \qquad (8)$$

$$\frac{pressure force}{Inertia force} = \frac{\left|\vec{F}_p\right|}{\left|\vec{F}_i\right|} \propto \frac{\Delta p L^2}{\rho V^2 L^2} = \frac{\Delta p}{\rho V^2} \qquad (9)$$

The term ρLV/μ is known as Reynolds number, ***Re***.

$Re \propto \frac{Inertia\ force}{Viscous\ force}$ is thus proportion to the magnitude ratio.

The term $\frac{\Delta p}{\rho V^2}$ is known as Euler number, ***Eu***.

∴ ***Re*** & ***Eu*** Represent the criteria of ***D.S.*** for the flows which are affected only by viscous, pressure and iertia forces. For example are

1- The full flow of fluid in a completely closed conduit
2- Flow of air past a low – speed aircraft
3- The flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, ***Re & Eu*** for a complete dynamic similarity between prototype and model must be the same for two. Thus

$$\frac{\rho_p\, L_p V_p}{\mu_p} = \frac{\rho_m\, L_m V_m}{\mu_m} \tag{10}$$

$$\frac{\Delta p_p}{\rho_p V_p^2} = \frac{\Delta p_m}{\rho_m\, V_m^2} \tag{11}$$

6- *D.S. of Flow Governed by Gravity and Inertia Forces.*

A flow of the type in which significant force are gravity and inertia forces, is found when a free surface is present. For example are

1- The flow of a liquid in an open channel.
2- The wave motion caused by the passage of a ship through water.
3- The flow over weirs and spillways.

The condition for ***D.S.*** of such flows requires

- The equality of ***Eu.***
- The equality of the magnitude ratio of gravity to inertia force at corresponding points in the system.

$$\frac{Gravity force}{Inertia force} = \frac{|\vec{F}_g|}{|\vec{F}_i|} \propto \frac{\rho L^3 g}{\rho V^2 L^2} = \frac{Lg}{V^2} \tag{12}$$

The reciprocal the term $\frac{(Lg)^{\frac{1}{2}}}{V}$ is known as Froude number, ***Fr***

$$\therefore\ Fr \ = \frac{V}{(Lg)^{\frac{1}{2}}}$$

∴ Dynamic similarity between prototype & model is the equality of Froude number

$$\frac{\sqrt{L_p\, g_p}}{V_p} = \frac{\sqrt{L_m\, g_m}}{V_m} \tag{13}$$

7- *D.S. of Flows with Surface Tension as the Dominant Force.*

Surface tension forces are important in certain classes of practical problems such as:

1- Flows in which capillary waves appear.
2- Flows of small jets and thin sheets of liquid injected by nozzle in air.
3- Flow of a thin sheet of liquid over a solid surface.

Dynamic similarity is the magnitude ratio

$$\frac{|\vec{F}_c|}{|\vec{F}_i|} \propto \frac{\sigma L}{\rho V^2 L^2} = \frac{\sigma}{\rho V^2 L} \tag{14}$$

The term $\frac{\sigma}{\rho V^2 L}$ is usually knows as Weber number, $\boldsymbol{Wb}$.

For dynamic similarity $(Wb\ \ )_m = (Wb\ \ )_p$

$$i.e., \frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p} \tag{15}$$

8- *D.S. of Flows with Elastic Force.*

The magnitude ratio of inertia to elastic force becomes

$$\frac{\text{Inertia force}}{\text{Elastic force}} = \frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2 L^2}{E L^2} = \frac{\rho V^2}{E} \tag{16}$$

The parameter $\frac{\rho V^2}{E}$ is known as Cauchy number.

For dynamic similarity flow ***(Cauchy)m = (Cauchy)p***

$$i.e., \frac{\rho_m V_m^2}{(E_s)m} = \frac{\rho_p V_p^2}{(E_s)_P} \tag{17}$$

If the flow is isentropic $E = E_s$ is isentropic bulk modulus of elasticity.

i = sound wave propagates through a fluid medium = $\sqrt{\frac{E_s}{\rho}}$

$\therefore$ *the term* $\rho V^2/E_s$ can be written as V^2/i^2

The ratio $\left(\frac{V}{i}\right) = \boldsymbol{Ma}$ is known as ***Mach number***, in the flow of air past high-speed aircraft, missiles, propellers and rotary compressors. In these cases equality of Mach number is a condition of dynamic similarity. Therefore

$$(Ma)_p = (Ma)_m$$

$$i.e \left(\frac{V_p}{i_p}\right) = \left(\frac{V_m}{i_m}\right) \tag{18}$$

Dimensionless terms	Representation magnitude ratio of the force	Name	Recommended symbol
$\rho LV/\mu$	$\frac{\text{Inertiaforce}}{\text{viscous force}}$	Reynolds number	Re
$\Delta p/\rho V^2$	$\frac{\text{pressureforce}}{\text{Inertiaforce}}$	Euler number	Eu
$V/(Lg)^{1/2}$	$\frac{\text{Inertia force}}{\text{Gravity force}}$	Froud number	Fr
$\frac{\sigma}{\rho V^2 L}$	$\frac{\text{surface tension}}{\text{Inertia force}}$	Weber number	Wb
$V/\sqrt{E_s/\rho}$	$\frac{\text{Inertiaforce}}{\text{Elastic force}}$	Mach number	Ma

Ex.1

When tested in water at 20C° flowing at 2 m/s, an 8-cm diameter sphere has a measured drag of 5 N. What will be the velocity and drag force on a 1.5m diameter weather balloon moored in sea-level standard air under dynamically similar condition?

Sol.

For water at 20C° $\rho \approx 998 \frac{kg}{m^3}$, & $\mu = 0.001 \frac{kg}{m.s}$

For air at sea level $\rho \approx 1.2255 \frac{kg}{m^3}$, $\mu = 1.78 * 10^{-5} \frac{kg}{m.s}$

The balloon velocity follows from dynamic similarity, which requires identical (***Reynolds number***).

$\boldsymbol{Re_m} = \boldsymbol{Re_p} = \frac{\rho VD}{\mu}$

$\mathrm{Re_m} = \frac{998*(2.0)(0.08)}{0.001} \approx 1.6 * 10^5 = \mathrm{Re_p}$

$1.6 * 10^5 = 1.2255 \frac{V_{ballon}(1.5)}{1.78*10^{-5}} ----\rightarrow V_{ballon} \approx 1.55 \frac{m}{s}$

Then the two spheres will be identical drag coefficients:

$$C_{D,m} = \frac{\Delta p}{\rho V^2} = \left[\frac{F}{\rho V^2\left(\frac{\pi d^2}{4}\right)}\right]_m = \left[\frac{F}{\rho V^2\left(\frac{\pi d^2}{4}\right)}\right]_p = \frac{F}{\rho V^2 d^2}$$

$$C_{D,m} = \frac{5}{998(2)^2(0.08)^2\frac{\pi}{4}} = 0.4986 = C_{D,p} = \frac{F_{balon}}{1.2255(1.55)^2(1.5)^2\pi/4}$$

Solve for $f_{ballon} \approx 1.296\ N$

Ex.2 A model of a reservoir having a free water surface within it is drained in 3 minutes by opening a sluice gate. The geometrical scale of the model is (1/100). How long would it take to empty the prototype?

Sol.

$$\frac{Q_m}{Q_p} = \frac{\frac{L_m^3}{T_m}}{\frac{L_p^3}{T_p}} = \frac{L_r^3}{T_r} \qquad\qquad T_r = \frac{T_m}{T_p}$$

The forces control the flow is

1- Gravity force $= \frac{(F_g)_m}{(F_g)_p} = \frac{(\rho g L^3)_m}{(\rho g L^3)_p} = \frac{W_m L_m^3}{W_p L_P^3} = W_r\, L_r^3$

2- Inertia force $= \frac{(F_i)_m}{(F_i)_p} = \frac{(m\ \ a)_m}{(m\ \ a)_p} = \frac{\rho_m}{\rho_p}\frac{L_m^3}{L_p^3}\left(\frac{L_r}{T_r^2}\right) = \rho_r\, L_r^3 \frac{L_r}{T_r^2}$

By equating the two ratio

$W_r\, L_r^3 = \rho_r\, L_r^3 \frac{L_r}{T_r^2}$

$\rho_r\, g_r\, L_r^3 = \rho_r\, L_r^3 \frac{L_r}{T_r^2}$

$T_r^2 = \frac{L_r}{g_r} ----\rightarrow \frac{T_m^2}{T_p^2} = \frac{\frac{1}{100}}{1};\quad since\ g_r = 1 \quad T_m = 3\ min$

$\therefore T_p = \sqrt{900} = 30\ \mathrm{min}$