Lecture-Two

Dimensional Analysis

1- *The Application of D.S and the Dimensional Analysis.*

A- *The concept.*

A physical problem may be characterized by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables. This gives a clue to reduction in the number of parameters requiring separate consideration in an experimental investigation.

Ex:- $Re = \frac{\rho V D_h}{\mu}$ Re 2000 ~ 4000 by varying V without change in any other independent dimensional variable .

In fact, the variation in the Re physically implies the variation in any of the dimensional parameters defining it.

B- *Dimensional Analysis.*

The dimensional analysis is a mathematical technique by which can be determining many dimensionless parameters and solving several engineering problems.

There are two existing approaches:-

1- Indicial method. (Method One)
2- Buckingham's ***pi*** theorem. (Method Two)

The dimensional analysis can be explain by the following,

- The Various physical quantities used in fluid phenomenon can be expressed in terms of fundamental quantities or primary quantities.
- Fundamental quantities are Mass (M), Length (L), Time (T), Temperature (θ) is used for compressible flow.
- The quantities which are expressed in terms of the fundamental or primary quantities are called derived or secondary quantities as (velocity, area, acceleration)
- The expression for a derived quantities in terms of the primary quantities is called the dimension of the physically quantities.
- A quantity may either be expressed dimensionally in M-L-T or F-L-T system.

Ex.1

Determine the dimensions of the following quantities.

(i) Discharge.
(ii) Kinematic viscosity.
(iii) Force.
(iv) Specific weight.

Sol.

(i) Discharge = area * velocity

$= L^2 * \frac{L}{T} = \frac{L^3}{T} = L^3T^{-1}$

(ii) Kinematic Viscosity(υ)=μ/ρ

Where (μ) given by (τ) = $\mu\frac{du}{dy}$

$$\mu = \frac{\tau}{du/_{dy}} = \frac{shearstress}{\frac{L}{T}\times\frac{1}{L}} = \frac{\frac{force}{Area}}{\frac{1}{T}}$$

$$\mu = \frac{mass\times acceleration}{Area\times\frac{1}{T}} = \frac{M\times\frac{L}{T^2}}{L^2\times\frac{1}{T}} = \frac{M\times L}{L^2T^2\times\frac{1}{T}}$$

$$\mu = \frac{M}{LT} = ML^{-1}T^{-1}$$

$$and\ \ \rho = \frac{mass}{volume} = \frac{M}{L^3} = ML^{-3}$$

$$\therefore \upsilon = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}$$

(iii) Force = mass * acceleration

$$= M * \frac{length}{Time^2} = \frac{ML}{T^2} = MLT^{-2}$$

(iv) Specific weight= Weight/volume= force/volume= $\frac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}$

2- *Buckingham's Pi Theorem. (Method-2)*

Assume, a physical phenomenon is described by

n = number of independent variables like x_1, x_2, x_3....x_n the phenomenon may be expressed as

$$F(x_1, x_2, x_3, \dots\dots x_n) = 0 \tag{1}$$

m = number of fundamental dimensions like mass, time, length and temperature or force, length, time and temperature.

Buckingham's theorem defining as the phenomenon can be described in terms of (***n-m***) independent dimensionless group like $\pi_1, \pi_2, \dots\dots\dots \pi_{n-m}$ where ***π*** terms, represent the dimensionless parameters and consist of different combinations of a number of dimensional variables out of the ***n*** independent variables. Therefore Eq.(1) can be reduced to

$$f(\pi_1, \pi_2, \dots\dots., \pi_{n-m}) = 0 \tag{2}$$

3- *Procedure for Determination π Terms.*

m = Number of fundamental dimensions like mass, (M), time (T), Length (L), temperature(θ)

n = number of independent variables or quantities included in physical problem such as (A_1,A_2,A_3-----A_n) where A_1,A_2,A_3-----A_n as pressure , viscosity and velocity, can also be expressed as

$$F1(A_1, A_2, A_3, ----A_n) = 0 \tag{3}$$

(***n-m***) = number of dimensionless parameter (π) like $\pi_1, \pi_2, \pi_3, \dots \pi_{n-m}$

π , is represent the dimensionless parameters and consist of different combinitions of a number of dimensional variable. Mathematically, if any variable A_1, depends on independent variable A_2.A_3…A_n the function $A_1 = F(A_2, A_3, \dots A_n)$

According to π-theorem, Eq. (3) can be written in terms of π- terms (dimensionless groups). Therefore the above equation can reduced to

$$F_1(\pi_1, \pi_2, \dots \pi_{n-m}) = 0 \tag{4}$$

The method of determining π parameters is

- Select (***m***) of the (A) quantities with different dimensions
- The above selection which contains among them (***m***) dimensions

- Using the (***m***) selection as repeating variables together with one of the other A quantities for each (π). Each π-term contains (***m+1***) variables.

Note-1, It is essential that no one of the ***m*** selected quantities used as repeating variable be derived from the other repeating variables.

Note-2, Let A_1,A_2,A_3 contain M,L and T, not necessarily in each one , but collectively.

Then the first π parameter is made up as

$$\left.\begin{aligned} &\pi_1 = A_2^{x1}\, A_3^{y1} A_4^{z1} A_1 \\ &\text{The second } \pi_2 = A_2^{x2} A_3^{y2} A_4^{z2} A_5 \\ &\text{And so on until } \pi_{n-m} = {A_2}^{x_{n-m}}\ {A_3}^{y_{n-m}}\, {A_4}^{z_{n-m}}\, A_n \end{aligned}\right\} \quad (5)$$

In a bove eqn's the exponents are to be determined

- The dimensions of (A) quantities are substituted
- The exponents of M,L and T are set equal to zero in π parameters

There produce three equations in three unknowns for each π parameter, so that the x,y,z exponents can be determined , and Hence the π parameter.

4- Selection of Repeating Variables (R.V).

1- (***m***) repeating variable must contain jointly all the fundamental dimension
2- The repeating variable must not form the non-dimensional parameter among them.
3- As far as possible, the dependent variable should not be selected as repeating variable
4- No two repeating variables should have the same dimensions
5- The repeating variables should be chosen in such a way that one variable contains geometric property (e.g , length , L, diameter, d, height, h) , other variable contains flow property (e.g velocity V, acceleration a) and the third variable contains fluid property (e.g mass density ρ, weight density W , dynamic viscosity μ)

(i) L,V,ρ
(ii) d,V,ρ
(iii) l, V,m
(iv) d,V,μ

Ex.2

Show that the lift force F_l on airfoil can be express as $F_L = \rho V^2\, d^2 \emptyset\, (\frac{\rho V d}{\mu}, \propto)$

Where ρ= mass density , V = velocity of flow
μ= dynamic viscosity ∝= Angle of incidence
d = A characteristic depth

Sol.

Left force F_L is function of;ρ, V, d, m, ∝ mathematically, $F_L = f(\ \mu, V, d, \rho, \propto) ----(i)$

Or $F_1(\,F_L, \rho\, V, d\, \mu, \propto) = 0 -------- (ii)$

∴ Total number of variable, we have ***n***=6

Writing dimensions of each variable

$F_L = M\, L\, T^{-2}, \rho =\ M\, L^{-3}, V = LT^{-1}, d = L, \mu = ML^{-1}T^{-1}\, , \propto = M^0 L^0 T^0$

Thus, number of fundamental dimensions, ***m***=3

$\therefore Number\ of\ \pi - terms = n - m = 6 - 3 = \mathbf{3}$

Eq. (ii) can be written as : $F_1(\pi_1, \pi_2, \pi_3) = 0 - - - - - - (iii)$

Each π-term contains (***m+1***) variables, where ***m***=3 and also equal to repeating variables (R.V). Choosing (d, V, ρ) as R.V

$\pi_1 = d^{a1}.V^{b1}.\rho^{c1}.F_L$

$\pi_2 = d^{a2}.V^{b2}.\rho^{c2}.\mu$

$\pi_3 = d^{a3}.V^{b3}.\rho^{c3}.\propto$

$\pi - term$:

$\pi_1 = d^{a1}.V^{b1}.\rho^{c1}.F_L$

$M^0L^0T^0 = L^{a1}(L\,T^{-1})^{b1}(ML^{-3})^{c1}(MLT^{-2})$

Equating the exponents of M,L,T respectively, we get

$for\ M: 0 = c_1 + 1 - - - - - \rightarrow\ c_1 = -1$

$for\ L: 0 =\ a_1 + b_1 - 3c_1 + 1$

$for\ T: 0 = -b_1\ - 2 - - - - \rightarrow\ b_1 = -2$

$\therefore\ a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Substituting the values of $a_1, b_1\ and\ c_1 in\ \pi_1$, we get

$\pi_1 = d^{-2}.V^{-2}.\rho^{-1}.F_L = \frac{F_L}{\rho V^2 d^2}$

$\pi_2 -\ term$:

$\pi_2 = d^{a_2}.V^{b_2}.\rho^{c_2}.\mu$

$M^0L^0T^0 =\ L^{a_2}.(LT^{-1})^{b2}.(ML^{-3})^{c_2}.(ML^{-1}T^{-1})$

Equating the exponents of M,L and T respectively , we get

$for\ M: 0 = c_2 + 1 - - - - \rightarrow\ c_2 = -1$

$FOR\ L: 0 = a_2 + b_2 - 3c_2 - 1$

For $T: 0 =\ -b_2 - 1 - - - - \rightarrow b_2 = -1$

$\therefore\ a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Substituting the values of $a_2, b_2, and\ c_2\ in\ \pi_2, we\ get$

$\pi_2 = d^{-1}.V^{-1}.\rho^{-1}.\mu = \frac{\mu}{\rho V d}$

$or\ \pi_2 = \frac{\rho V d}{\mu}$

$\pi_3 -\ term$

$\pi_3 =\ d^{a_3}.V^{b_3}.\rho^{c_3}.\propto$

$M^0L^0T^0 =\ L^{a_3}.(LT^{-1})^{b_3}.(ML^{-3})^{c_3}.(M^0L^0T^0)$

Equating the exponents of M, L and T respectively , we get

$for\ M: 0 =\ c_3 + 0 - - - - \rightarrow\ c_3 = 0$

$for\ L: 0 =\ a_3 + b_3 - 3c_3 + 0$

$for\ T: 0 =\ -b_3 + 0 - - - - \rightarrow b_3 = 0$

$\therefore\ a_3 = 0$

$\therefore\ \pi_3 = d^0.v^0\rho^0.\propto = \propto$

Substituting the values of $\pi_1, \pi_2\ and\ \pi_3\ in\ Eq.(iii), we\ get$

$f_1\left(\frac{F_L}{\rho V^2 d^2}, \frac{\rho V d}{\mu}, \propto\right) = 0$

$\frac{F_L}{\rho V^2 d^2} = \emptyset\left(\frac{\rho V d}{\mu}, \propto\right)$

$or \quad F_L = \rho V^2 d^2 \emptyset\left(\frac{\rho V d}{\mu}, \propto\right)$

Ex.3

The discharge through a horizontal capillary tube is thought to depend upon the pressure drop per unit length, the diameter and the viscosity. Find the form of the discharge equation.

Sol.

Quantity	Dimensions
Discharge Q	$L^3\ T^{-1}$
Pressure drop/length Δp/l	$M\ L^{-2}T^{-2}$
Diameter D	L
Viscosity μ	$M\ L^{-1}T^{-1}$

Then $F\left(Q, \frac{\Delta p}{L}, D, \mu\right) = 0$

Three dimension are used, and with four quantities these will be one π parameter

$\boldsymbol{n} = 4, \boldsymbol{m} = 3 - - - - \rightarrow \pi = n - m = 4 - 3 = 1$

$\pi = Q^{X_1}(\frac{\Delta p}{l})^{Y_1} D^{Z_1} \mu$

Substituting in the dimension gives

$\pi = (L^3\ T^{-1})^{X_1}\ (ML^{-2}T^{-2})^{Y_1}\ (L^{Z_1})(M\ L^{-1}T^{-1}) = M^0 L^0 T^0$

The exponents of each dimension must be the same on both sides of the equation.

With L; $3X_1 - 2Y_1 + Z_1 - 1 = 0 - - - - - - -(i)$

M; $Y_1 + 1 = 0 - - - - \rightarrow Y_1 = -1$

T; $-X_1 - 2Y_1 - 1 = 0 - - - - - - - \rightarrow X_1 = 1$

From Eq. (i) $Z_1 = -4$

$\pi = Q \frac{\mu}{\left(\frac{\Delta p}{l}\right) D^4}$

After solving for Q

$Q = C \frac{\Delta p}{L} \frac{D^4}{\mu}$

Ex.4

Consider pressure drop in a tube of length l, hydraulic diameter d, surface roughness ∈, with fluid of density ρ and viscosity μ moving with average velocity U. Using Buckingham's π - theorem obtain an expression for Δp .

Sol.

This can be expressed as

$f(\Delta p, U, d, l, \in, \rho, \mu) = 0$

Now $\boldsymbol{n}$=7 since the phenomenon involves 7 independent parameters.

We select ρ, U, d as repeating variables (so that all 3 dimensions are represented)
Now, 4 π ----→(7-3) parameters are determined as

$\pi_1 = \rho^{a_1} U^{b_1} d^{c_1} \Delta p$
$\pi_2 = \rho^{a_2} U^{b_2} d^{c_2} \mu$
$\pi_3 = \rho^{a_3} U^{b_3} d^{c_3} l$
$\pi_4 = \rho^{a_4} U^{b_4} d^{c_4} \in$

Now basic units

$\rho - - - - -\rightarrow ML^{-3}$
$U - -\rightarrow LT^{-1}$
$d - - -\rightarrow L$
$\Delta p - - -\rightarrow ML^{-1}T^{-2}$
$\mu - - - - -\rightarrow ML^{-1}T^{-1}$
$\in - - -\rightarrow L$
$l - - -\rightarrow .L$

All π parameters $- - - -\rightarrow M^0 L^0 T^0$

$a_1 = -1;\ b_1 = -2\ ;\ C_1 = 0$
$a_2 = -1;\ b_2 = -1\ ;\ c_2 = -1$
$a_3 = 0;\ b_3 = 0;\ c_3 = -1$
$a_4 = 0;\ b_4 = 0;\ c_4 = -1$

Thus writing $\pi_1 = f(\pi_2, \pi_3, \pi_4)$
∴ The π group can be written as follows,

$\pi_1 = d^0 V^{-2} \rho^{-1} \Delta p = \frac{\Delta p}{\rho V^2}$ $\quad$ $\boldsymbol{Eu\ \ .No.}$

$\pi_2 = \frac{\mu}{dV\rho}\ or\ \frac{dV\rho}{\mu}$ $\quad$ $\boldsymbol{Re\ \ .N_o.}$

$\pi_3 = d^{-1} V^0 \rho^0 l = \frac{l}{d}$

$\pi_4 = \frac{\in}{d}$

∴ The new relation can be writing

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{dV\rho}{\mu}, \frac{l}{d}, \frac{\in}{d}\right) = 0$$

When conclude Δp

$$\frac{\Delta p}{\rho V^2} = f_2\left(Re\ \ , \frac{l}{d}, \frac{\in}{d}\right)$$

$$\rho = \frac{W}{g}$$

$$\frac{\Delta p}{W} = \frac{V^2}{2g} . f_2\left(Re\ \ , \frac{l}{d}, \frac{\in}{d}\right)$$

The pressure drop is function of (L/d) exponent to(1) in darcy equation

$$\frac{\Delta p}{W} = \frac{V^2}{2g} . \frac{L}{d} . f_3\left(Re\ \ , \frac{\in}{d}\right)$$

Therefore

$$\frac{\Delta p}{W} = (Factor\ \boldsymbol{f})\left(\frac{L}{d}\left(\frac{V^2}{2g}\right)\right)$$

Ex.5

Assume the input power to a pump is depend on the fluid weight per unit volume, flow rate and head produced by the pump. Create a relation by dimensional analysis between the power and other variables by two methods.

Method-2

$F(P, W, Q, H) = 0$

The variables in dimensions are

$P ---\to FLT^{-1}$

$Q --\to L^3T^{-1}$

$W --\to FL^{-3}$

$H ---\to L$

The four variables in 3 fundamental dimensional $\therefore$ $\pi\ group\ is (4-3) = 1$

Choice Q, W, H as variable with unknown exponent

$\therefore\ \pi_1 = (Q)^{a_1}\,(W)^{b_1}\,(H)^{c_1}\,P$

$or\ \pi_1 = (L^{3a_1}\,T^{-a_1})(F^{b_1}\,L^{-3b_1})(L^{c_1})(F\,L\,T^{-1}) = F^0L^0T^0$

Exponent equality foe F,L,T producing

$a_1 = -1, b_1 = -1, c_1 = -1$

$\therefore\ \pi_1 = Q^{-1}\,W^{-1}H^{-1}\,P = \frac{P}{QWH}$

$\pi_1 = F\left(\frac{P}{QWH}\right)$

$P = K(QWH)$

Ex.6

Assume the input power to a pump is depend on the fluid weight (W), flow rate (Q) and head produced by pump (H), create a relation by dimension analysis between the power input and other variables by using FLT system.

Sol.

Step-1

Quantities	Dimensions
Power (P)	$F\,L\,T^{-1}$
Flow rate (Q)	L^3T^{-1}
Weight(W) per unit volume	$F\,L^{-3}$
Head (H)	L

Step-2:- there are four variables & (3) fundamental dimensions

$\therefore\ \pi\ group\ is\ (4-3) = 1$

Step-3:

Choice Q, W, H as variable with unknown exponent.

$F_1(P,W,Q,H) = 0 \longrightarrow F_2(\pi_1) = 0$

$\therefore\ \pi_1 = (Q)^{a_1}\,(W)^{b_1}(H)^{c_1}\,P$

Or $\pi_1 = (L^3T^{-1})^{a_1}\,(F\,L^{-3})^{b_1}(L)^{C_1}\,(FLT^{-1})^1 = F^0L^0T^0$

F] $b_1 + 1 = 0 -\!\!-\!\!\rightarrow\ b_1 = -1$

L] $3a_1 - 3b_1 + c_1 + 1 = 0 \quad \rightarrow 3a_1 + c_1 = -4$

T] $-a_1 - 1 = 0 -\!\!-\!\!\rightarrow\ a_1 = -1$

$\therefore\ c_1 = -1$

$\therefore\ \pi_1 = Q^{-1}W^{-1}H^{-1}\,P = \frac{P}{QWH}$

$\pi_1 = f\left(\frac{P}{QWH}\right)$

$P = K(Q,W,H)$

Ex.7

Assuming the resistant force for a body submerged in a fluid is function of (density ρ , velocity V, viscosity μ and characteristic length L). Conclude a general equation of resistant force by using FLT system.

Sol.

Step.1

Quantities	Dimension
Force(F)	F
Density(ρ)	$F\,L^{-4}T^2$
Velocity(V)	$L\,T^{-1}$
Viscosity(μ)	$F\,L^{-2}T$
Length(L)	L

$F_1(F,\rho,V,L,\mu) = 0 \quad n = 5\,, m = 3$

Step-2

We have 5 variables with 3 fundamental dimensions

$\therefore\ \pi\ groups = 5 - 3 = 2$

We choice 3 repeated variables of unknown exponents

$\pi_1 = (L)^{a_1}\,(V)^{b_1}\,(\rho)^{c_1}\,F = (L)^{a_1}\,(LT^{-1})^{b_1}\,(F\,T^2L^{-4})^{c_1}\,(F) = F^0L^0T^0$

F] $C_1 + 1 = 0 -\!\!-\!\!\rightarrow\ C_1 = -1$

L] $a_1 + b_1 - 4c_1 = 0 -\!\!-\!\!\rightarrow\ a_1 + b_1 = -4$

T] $-b_1 + 2C_1 = 0 \longrightarrow\ b_1 = -2;\ \therefore\ a_1 = -2$

$\therefore\ \pi_1 = L^{-2}\,V^{-2}\rho^{-1}F = F/L^2V^2\rho$

$\pi_2 = (L)^{a_2}\,(V)^{b_2}\,(\rho)^{C_2}\,\mu = (L)^{a_2}\,(L\,T^{-1})^{b_2}\,(F\,L^{-4}T^2)^{C_2}\,(F\,L^{-2}T) = F^0L^0T^0$

F] $C_2 + 1 = 0 -\!\!-\!\!\rightarrow\ C_2 = -1$

L] $a_2 + b_2 - 4C_2 - 2 = 0$

$a_2 + b_2 + 2 = -\!\!-\!\!\rightarrow\ a_2 + b_2 = -2$

T] $-b_2 + 2C_2 + 1 = 0 --\rightarrow\ b_2 = -1$

$\therefore\ a_2 = -1$

$\therefore\ \pi_2 = L^{-1}V^{-1}\rho^{-1}\mu = \frac{\mu}{LV\rho} --\rightarrow \pi_2^{-1} = R_e$

$\therefore f(\pi_1, \pi_2) = 0$

$F_1(\pi_1, \pi_2^{-1}) = 0 --\rightarrow\ F_1\left(\frac{F}{L^2V^2\rho}, R_e\right) = 0$

$\therefore F = L^2V^2\rho\, F_2(R_e)$