Lecture-Three

Laminar Flow in Different Shapes

1- Introduction.

Real fluids possess viscosity, while ideal fluid is inviscid. The viscosity of fluid introduce resistance to motion by developing shear or frictional stress between the fluid layers and between fluid layers and the boundary, which causes the real fluid to a adhere to the solid boundary and hence no relative motion between fluid layer and solid boundary.

Viscosity causes the flow to occur in two modes laminar and turbulent flow. Reynolds number < 2000, the flow is always laminar through a pipe which is critical value of ***Re*** for circular pipe. Flow between parallel plates based on mean velocity and distance between the plates.

$$Re \quad = \frac{Inertia force}{viscous force}$$

The flow is laminar when one of the conditions occurs

i) Viscosity is very high.
ii) Velocity is very low.
iii) The passage is very narrow.

2- Relationship between Shear Stress and Pressure Gradient.

The shear stress is maximum at the boundary and gradually decreases with increase in distance from the solid boundary where the velocity is zero at the boundary. A pressure gradient exists which overcome the shear resistance and causes the fluid to flow. Due to non-uniform distribution of velocity, the fluid at any layer moves at a higher velocity than the layer below.

The motion of the fluid element will be resisted by shearing or frictional force which must be overcome by maintaining a pressure gradient in the direction of flow, from Fig (1), Let τ = shear stress on the lower face ABCD of the element

$\tau + \frac{\partial \tau}{\partial y}\delta y =$ Shear stress on the upper face $\acute{A}\acute{B}\acute{C}\acute{D}$ of the element.

For 2-dimensional steady flow there will be no shear stress on the vertical faces ABB'A' & CDD'C' as in Fig. (1). Thus the only forces acting on the element in the direction of flow (x-axis) will be the pressure and shear forces. Let $\delta x, \delta y\ and\ \delta z$ are element thickness in x, y and z directions.

Net shearing force on the element in y-direction is equal to

$$= \left(\tau + \frac{\partial \tau}{\partial y}\ \delta y\right)\delta x.\delta z - \tau.\delta x \delta z = \frac{\partial \tau}{\partial y}\delta x.\delta y.\delta z \qquad (1)$$

Net pressure force on the element in x-direction is equal to

$$= \ p.\delta y.\delta z - \left(p + \frac{\partial p}{\partial x}\delta x\right)\delta y.\delta z = -\frac{\partial p}{\partial x}\delta x.\delta y.\delta z \qquad (2)$$

For the flow to be steady and uniform these begin no acceleration, the sum of the forces must be zero, from Eq's (1 & 2)

$$\frac{\partial \tau}{\partial y}.\delta x.\delta y.\delta z - \frac{\partial p}{\partial x}.\delta x.\delta y.\delta z = 0$$

The relationship between shear stress and pressure gradient is

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y} \qquad (3)$$

Eq. (3) indicates that the pressure gradient in the direction of flow is equal to the shear gradient in the direction normal to the direction of flow. This is applicable for laminar and turbulent flow.

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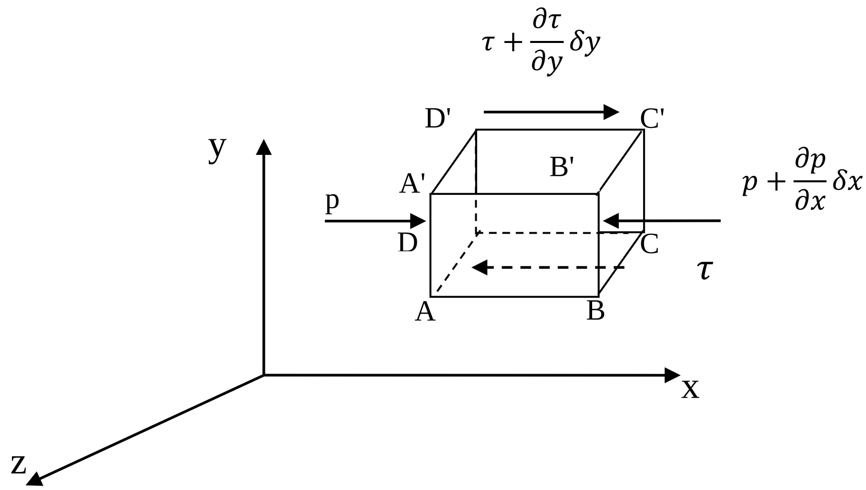


Figure 1: Pressure and Shear stress Forces on a Fluid Element.

3- ***Laminar Flow between Parallel Plates.***

Consider two parallel plates with (h) distance apart. For steady flow between them a pressure gradient $\partial p/\partial x$ exist which related shear stress in y-direction.

Since $\tau = \mu \frac{du}{dy}$

Then $\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$

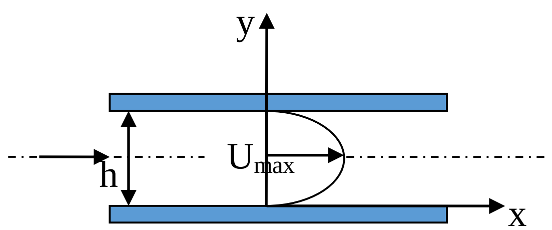


Figure 2: Laminar flow between parallel plates.

Integration gives

$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + C_1 y + C_2$

$u = 0 \;\; at\; y = 0\; and\; y = h$

$C_2 = 0\,, \quad C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h$

And

$u = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)$

This equation is a parabola shape with vertex at center line (y=h/2) when the maximum velocity occurs

$U_{max} = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{h^2}{2} - \frac{h^2}{4}\right)$

Or$u_{max} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) h^2$ (4)

The (-ve) of pressure gradient is the pressure drop in the direction of flow. The discharge ***dq*** through a small area of depth ***dy*** per unit width is

dq = udy

$dq = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)dy$

$Q = \int_0^h \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy \quad - y^2)dy$

$Q = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\left(h\frac{y^2}{2} - \frac{y^3}{3}\right)\right)_0^h$

$Q = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right) h^3$ (5)

Mean velocity of flow $\bar{u} = \frac{Q}{flow\ area} = \frac{Q}{h*1}$

$\bar{u} = \frac{\frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)h^3}{h} = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right) h^2$ (6)

The above velocity ($\bar{u}$) may be used to calculate the pressure drop

$-\partial p = \frac{12\mu\bar{u}}{h^2}\partial x \quad \rightarrow \quad -\frac{\partial p}{\partial x} = \frac{12\mu\bar{u}}{h^2}$ (7)

From Eq's (4 & 6)

$U_{max} = \frac{3}{2}\bar{u}$

The pressure drop between two sections with distance x_1 *and* x_2 from origin is

$\int_{p_1}^{p_2}(-dp) = \int_{x_1}^{x_2}\frac{12\mu\bar{u}}{h^2}dx$

$p_1 - p_2 = \frac{12\mu\bar{u}}{h^2}(x_2 - x_1)$

If L is the length between the sections

$p_1 - p_2 = \frac{12\mu\bar{u}L}{h^2}$ (8)

The variation of shear stress in the y-direction is

$\tau = \mu\frac{\partial}{dy}\left[\frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)\right]$

$\tau = \frac{1}{2}\left(-\frac{\partial p}{\partial x}\right)(h - 2y)$

$\tau = -\frac{\partial p}{\partial x}\left(\frac{h}{2} - y\right)$ (9)

Shearing stress varies linearly with y, it is maximum at the boundary, y=0 and y=h

$at\ y = 0 \quad \tau_0 = \frac{h}{2}\left[-\frac{\partial p}{\partial x}\right] = \frac{h}{2}\left(\frac{12\mu}{h^2}\bar{u}\right) = \frac{6\mu\bar{u}}{h}$ (10)

$At\ y = h \quad \tau_0 = -\frac{h}{2}\left(-\frac{\partial p}{\partial x}\right) = -\frac{h}{2}\left(\frac{12\mu}{h^2}\bar{u}\right) = -\frac{6\mu\bar{u}}{h}$ (11)

Shearing stress at the center y = h/2 is zero

$\tau = -\frac{\partial p}{\partial x}\left(\frac{h}{2} - \frac{h}{2}\right) = 0$

Ex.1

Incompressible fluid flows through a rectangular passage of width (b), small depth (t) and length L, in the direction of flow. If the pressure drop between the two ends is p calculate the shear stress at the wall of the passage in terms of mean velocity and the coefficient of viscosity

Sol.

$p = \frac{12\mu\bar{u}L}{t^2}$

$\tau_0 = \frac{t}{2}\left(-\frac{\partial p}{\partial x}\right) = \frac{tp}{2L}$

Then $\tau_0 = \frac{6\mu\bar{u}}{t}$

Ex.2

Water at 20C° flows between two large parallel plates separated by a distance of 16mm. calculate

i) Max. velocity

ii) Shear stress at the wall if the average velocity is (0.4 m/s) (take μ for water = 0.01 Poise)

Sol.

i) $U_{max} = \frac{t^2}{8\mu}\left(-\frac{\partial p}{\partial x}\right) = \frac{3}{2}\bar{u} = \frac{3}{2} * 0.4 = 0.6\ \frac{m}{s}$

ii) Shear stress at the wall

$$\tau_0 = -\frac{6\mu\bar{u}}{t} = -0.15\ N/m^2$$

4- *Couette Flow.*

Couette flow is the flow between two parallel plates as in Fig. (3) one plate is at rest and the other is moving with a velocity U, assuming infinitely large in z-direction

The governing Equation is

$$\frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$$

Flow is independent of any variation in z-direction the boundary condition are

i) At y = 0 , u = 0

ii) At y = h , u = U

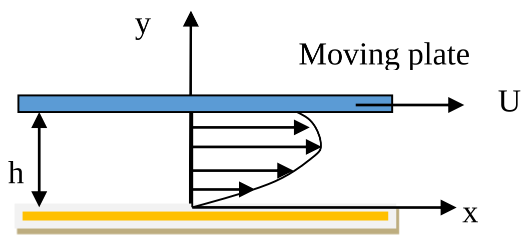


Figure 3: Couette flow between parallel plates

After integration twice we get

$$u = \frac{1}{2\mu}\frac{dp}{dx}\,y^2 + c_1y + c_2$$

At y = 0, u = 0, then $c_2 = 0$

$$u = \frac{1}{2\mu}\frac{dp}{dx}\,y^2 + c_1y$$

At y=h , u= U

$$\therefore\ C_1 = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}\,h$$

Then the expression for ***u*** becomes

$$u = \frac{y}{h}\,U - \frac{1}{2\mu}\frac{dp}{dx}\,(hy - y^2) \qquad (12)$$

Multiply and divided by(h^2)

$$\text{Or } u = \frac{y}{h}\,U - \frac{h^2}{2\mu}\,.\frac{dp}{dx}.\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (13)$$

Eq. 13 can also be expressed in the form

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U}.\frac{dp}{dx}\,.\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

$$\text{Or } \frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (14)$$

Where $P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right)$

P is known as the non- dimensional pressure gradient. When P=0, the velocity distribution across the channel reduced to

$\frac{u}{U} = \frac{y}{h}$ *is known as simple couette flow .*

- When P>0 , i.e for a negative pressure gradient (-dp/dx) in the direction of motion , the velocity is positive over the whole gap.
- When P < 0, these is positive or adverse pressure gradient in the direction of motion and the velocity over a portion of channel width can become negative and back flow may occur near the wall, which is at rest

5- *Maximum and Minimum Velocities.*

The variation of maximum and minimum velocity in the channel is found out by setting du/dy = 0 from Eq. (14), we can write

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right)$$

Setting du/dy = 0 gives

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \qquad (15)$$

By studying Eq. (15) we conclude that

1- The maximum velocity for P=1 occurs at y/h =1 and equal to U. For P>1 , the maximum velocity occurs at a location y/h <1.
2- i.e that with P>1, the fluid particles attain a velocity higher than that of the moving plate.
3- for P= -1 , the minimum velocity occurs at y/h =0 for P< -1 , the minimum velocity occurs at a location y/h>1
4- This means that these occurs a back flow near the fixed plate. The values of maximum and minimum velocities can be determined by substituting the value of y from Eq.(15) into Eq. (14) as

$$\left.\begin{aligned} U_{max} &= \frac{U(1+P)^2}{4P} \text{ for P} \geq 1 \\ U_{min} &= \frac{U(1+P)^2}{4P} \ for\ P \leq 1 \end{aligned}\right\} \quad (16)$$

The expression for shear stress can be obtained by substituting the value of ***u*** in Newton's equation of viscosity

$$\tau = \mu \frac{du}{dy} = \mu \frac{d}{dy}\left[\frac{yU}{h} - \frac{1}{2\mu}\frac{dp}{dx}(hy - y^2)\right]$$

$$\tau = \mu \frac{U}{h} + \left(-\frac{dp}{dx}\right)\left(\frac{h}{2} - y\right) \quad (17)$$

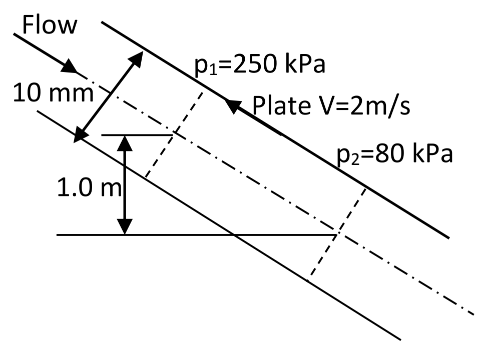
The shear stress at the center is

$$\tau = \mu \frac{U}{h} \quad (18)$$

Ex.3

Laminar flow takes place between parallel plates 10 mm apart. The plates are inclined at 45° with the horizontal. For oil of viscosity 0.9 kg/m.s and mass density is 1260 kg/m^3, the pressure at two points 1.0 m vertically apart are 80 kN/m^2 and 250 kN/m^2 when the upper plate moves at 2.00 m/s velocity relative to the lower plate but in opposite direction to flow determine

i) velocity distribution
ii) max. velocity
iii) shear stress on the top plate



Sol.

Consider section 1&2 from Bernoulli's Eqn.

$$\mathrm{H}_1 - \mathrm{H}_2 = -\left(\frac{p_1}{\gamma} + Z_1\right) + \left(\frac{p_2}{\gamma} + Z_2\right)$$

$$= -\left(\frac{250000}{9.806*1260} + 1\right) + \left(\frac{80000}{9.806*1260} + 0\right)$$

$$H_1 - H_2 = -21.234 + 6.475 = -14.759m\ in\ \ 1.414m\ length$$

Since $H_1$is greater than H_2, flow will be in down word direction.

$$\frac{\partial H}{\partial x} = -\frac{14.759}{1.414} = -10.438$$

And $\frac{\partial p}{\partial x} = \gamma \frac{\partial H}{\partial x} = -10.438 * 1260 * 9.806 = -128.97\ \frac{kN/m^2}{m}$

$$\frac{u}{U} = \frac{y}{h} - \frac{1}{2\mu U}\frac{\partial p}{\partial x}(yh - y^2)$$

$$U = -2\frac{m}{s},\ h = 0.01m, \mu = \frac{0.9kg}{ms}$$

$$\therefore u = -\frac{2}{0.01}y - \frac{1}{2*0.9}\ (-128967.33)(0.01y - y^2)$$

i) $u = 516.4364\, y - 71648.5y^2$

To find y at which u is max. set du/dy=0 $= 516.486 - 143297.2y$ or $y = 3.604 * 10^{-3}m$

ii) $\therefore\ u_{max} = (516.486 * 0.003604) - (71648.2 * 0.003604^2) = 0.9308\,\frac{m}{s}$

$$\tau_0 = \mu\,(\frac{\partial u}{\partial y})_{y=0.01} = 0.9(516.486 - 143297.2 * 0.01) = -824.837\ N/m^2$$