Lecture-Four

Laminar Flow in Pipe of Circular Cross-Section.

1- Hagen-Poiseuille Flow.

Consider fully developed laminar flow through a straight tube of circular cross – section as in Fig.(1). Rotational symmetry is considered to make the flow two – dimensional axisymmetry. Let us take ***x-axis*** as the axial of the tube along which all the fluid particles travel, i.e.

$V_x \neq 0, V_r = 0, V_\theta = 0$

Now from continuity equation, we obtain

$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_x}{\partial x} = 0 \left[for\ rotational\ symmetry, \frac{1}{r}.\frac{\partial v_\theta}{\theta} = 0\right]$

This means $V_x = V_x(r,t)$

Invoking $\left[V_r = 0, V_\theta = 0\ \frac{\partial V_x}{\partial x} = 0, and\ \frac{\partial}{\partial \theta}(\ any\ quantitng) = 0\right]$

With Navier–Stokes equation, we obtain in the x-direction

$$\frac{\partial V_x}{\partial t} = -\frac{1}{\rho}.\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}.\frac{\partial V_x}{\partial r}\right) \qquad (1)$$

For steady flow, the governing equation becomes

$$\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}.\frac{\partial V_x}{\partial r} = \frac{1}{\mu}\frac{dp}{dx} \qquad (2)$$

The boundary conditions are

i) At $r = 0, V_x\ is\ finit\ \& \frac{\partial V_x}{\partial r} = 0$

ii) At r = R, V_x=0 yield Eq.(2) can be written after multiplying by ***r***

$r\frac{d^2 V_x}{dr^2} + \frac{dV_x}{dr} = \frac{1}{\mu}.\frac{dp}{dx}r$

$or\ \frac{d}{dr}\left(r\frac{dV_x}{dr}\right) = \frac{1}{\mu}\frac{dp}{dx}\ r$ by integration

$r\frac{dV_x}{dr} = \frac{1}{2\mu}.\frac{dp}{dx}\ r^2 + A$

$\frac{dV_x}{dr} = \frac{1}{2\mu}.\frac{dp}{dx}\ r + \frac{A}{r}$ by integration

$V_x = \frac{1}{4\mu}.\frac{dp}{dx}\ r^2 + A\ln r + B$

At $r = 0\ V_x = finite\ \& \frac{dV_x}{dr} = 0 \rightarrow A = 0$

$at\ r = R, V_x = 0$

$B = -\frac{1}{4\mu}.\frac{dp}{dx}.R^2$

$$\therefore\ V_x = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)\left[1 - \frac{r^2}{R^2}\right] \qquad (3)$$

This shows that the axial velocity profile in a fully developed laminar pipe flow is having parabolic variation along ***r***.

At $r = 0, as\ such, V_x = V_{x\,max}$

$$V_{x\,max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right) \qquad (4)$$

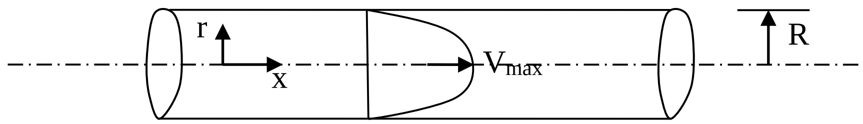

Figure 1: Flow in circular pipe.

2- Volumetric Flow Rate.

The average velocity in pipe is

$$V_{av.} = \frac{Q}{\pi R^2} = \frac{\int_0^R 2\pi r\, V_x(r) dr}{\pi R^2} \text{ substitute Eq. 3}$$

$$or\ V_{av.} = \frac{\frac{2\pi R^2}{4\mu}\left(-\frac{dp}{dx}\right)\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]}{\pi R^2}$$

$$V_{av.} = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = \frac{1}{2} V_{x\,\max} \quad \rightarrow V_{x\,max} = 2V_{av} \qquad (5)$$

Now, the discharge ***Q*** through a pipe is given by

$$Q = \pi R^2 V_{av} \qquad (6)$$

$$Q = \pi R^2 \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right)$$

$$or\ Q = -\frac{\pi d^4}{128\mu}\left(\frac{dp}{dx}\right) \qquad (7)$$

From Eq's 4 & 5

$$\frac{p_1 - p_2}{L} = 4\, V_{max} \frac{\mu}{R^2} = 32\mu \frac{V_{av}}{d^2} \qquad (8)$$

Eq. 8 is known as the ***Hagen- Poiseuille*** equation.

Ex.1

Oil mass density is 800 kg/m^3 and dynamic viscosity is 0.002 kg/m.s flow through 50mm diameter, pipe length is 500 m and the discharge flow rate is 0.19*10^{-3} m^3/s determine

i) Reynolds number of flow.
ii) Center line velocity.
iii) Loss of pressure in 500 m length.
iv) Pressure gradient.
v) Wall shear stress.

Sol.

$$V_{av.} = \frac{4Q}{\pi d^2} = \frac{4*0.19*10^{-3}}{\pi*(0.05)^2} = 0.0968\,\frac{m}{s}$$

i) $R_e = \frac{Vd\rho}{\mu} = \frac{0.0968*0.05*800}{0.002} = 1936.0$

ii) $V_{x\,max} = 2V_{av.} = 2 * 0.0968 = 0.1936\,\frac{m}{s}$

iii) From Eq. 7.26

$$\frac{p_1 - p_2}{L} = 4\, V_{max} \frac{\mu}{R^2} = 32\mu \frac{V_{av}}{d^2}$$

$$\therefore\ p_1 - p_2 = \frac{32\mu V_{av} L}{d^2} = \frac{32*0.002*0.0968*500}{(0.05)^2} = 1239.04\,\frac{N}{m^2}$$

iv) $\frac{dp}{dL} = \frac{p_1 - p_2}{L} = \frac{1239.04}{500} = \frac{2.478\frac{N}{m^2}}{m} = 2.478\ Pa/m$

v) $\tau_0 = \frac{(p_1 - p_2)d}{4L} = (1239.04) * \frac{0.05}{4*500} = 0.03098\,\frac{N}{m^2}, Eq. 10$

3- Shear Stress in Horizontal Pipe.

A force balance for steady flow in horizontal pipe as in Fig. 2 yields

$$p_1(\pi r^2) - p_2(\pi r^2) - \tau(2\pi r L) = 0$$

$$or\ \tau = \frac{(p_1 - p_2) r}{2L} \qquad (9)$$

From Eq. 9

$at\ r = 0\ \tau = 0$

$r = R \quad \tau = \tau_0$

$\tau_0 = \frac{(p_1 - p_2)d}{4L}$ (10)

Eq. 9 is valid for laminar & turbulent flow.

$\left(\frac{p_1 - p_2}{\rho g}\right)$ Represent the energy drop per unit weight (h_L) multiply Eq.9 by (ρg/ρg) yields

$\tau = \frac{\rho g r}{2L}\left(\frac{p_1 - p_2}{\rho g}\right) = \frac{\rho g h_L}{2L} r$ (11)

$\therefore\ h_L = \frac{2\tau_0 L}{\rho g R} = \frac{4\tau_0 L}{\rho g d}$ (12)

$\tau = \tau_0\ at\ r = R$

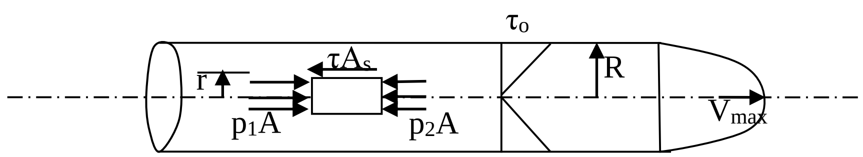

Figure 2: Forces on element in horizontal pipe.

4- Shear Stress in Inclined Pipe.

The energy equation may be written in pipe when related the loss to available energy reduction as in Fig.(a)

$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_{f_{1-2}}$

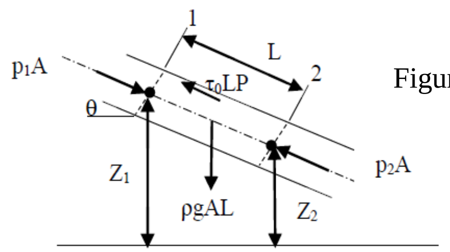

Figure (a)

Since the velocity head $\left(\frac{V^2}{2g}\right)$ is the same

$h_f = \frac{p_1 - p_2}{\rho g} + z_1 - z_2$ (13)

$\therefore\ h_f = \frac{\Delta p}{\rho g} + \Delta z$ (14)

Applying the linear – momentum eqn. in the L-direction

$\sum F_l = 0 = (p_1 - p_2)A + \gamma AL\ sin\theta - \tau_0 LP = \dot{m}(V_2 - V_1) = 0$

(P) is the wetted perimeter of the conduit ,i.e , the portion of the perimeter where the wall is in contact with the fluid when the conduit not circular pipe.

$L\ sin\theta = z_1 - z_2$

$\frac{p_1 - p_2}{\rho g} + z_1 - z_2 = \frac{\tau_0 LP}{\rho g A}$ (15)

From Eq. 13& 15

$h_f = \frac{\tau_0 LP}{\rho g A}$ (16)

From experiment

$\tau_0 = \lambda \frac{\rho}{2} V^2$ (17)

$\therefore\ h_f = \lambda \frac{\rho}{2} V^2 \frac{LP}{\gamma A} = \lambda \frac{L}{R} \frac{V^2}{2g}$ (18)

R_h=A/P

R_h= hydraulic Radius of the conduit

For a pipe R_h=d/4 ; λ=f/4

Where λ is the non-dimensional factor, the h_f head loss due to friction can be written as follows,

$\therefore\ h_f = \frac{f}{4} \frac{L}{d} \frac{4}{} \frac{V^2}{2g} = f \frac{L}{d} \frac{V^2}{2g}$ (19)

Eq. 19 is the Darcy – Weisbach equation, valid for duct flows of any cross-section and for laminar and turbulent flow, ***f*** is the friction factor ***f***=4 λ

By equating Eq's 12 & 19

$\frac{4\tau_0 L}{\rho g d} = f \frac{L}{d} \frac{V^2}{2g}$

$\therefore\ \tau_0 = \frac{f \rho V^2}{8}$ (20)

In Hagen-Poiseuille eqn.

$V_{av} = \frac{\Delta p d^2}{32 \mu L}$ From Eq. 8

$\Delta p = \rho g\ h_f ----\rightarrow\ h_f = \frac{\Delta p}{\rho g}$

$\therefore V_{av} = \frac{\rho g\ h_f d^2}{32 \mu L}$

$h_f = \frac{32 V_{av} \mu L}{\rho g d^2} = f \frac{L}{d} \frac{V^2}{2g}$

$= \left(\frac{64 V_{av} \mu L}{2 \rho g d^2} \right) = \frac{64}{\frac{\rho d V_{av}}{\mu}} \frac{L}{d} \frac{V_{av}{}^2}{2g} = \frac{64}{R_e} \frac{L}{d} \frac{V_{av}{}^2}{2g}$

$h_f = f \frac{L}{d} \frac{V_{av}{}^2}{2g} = \frac{64}{R_e} \frac{L}{d} \frac{V_{av}{}^2}{2g}$ (21)

$\therefore f = \frac{64}{Re}$ (22)

It applies to all roughness and may be used for the solution of laminar flow problems in pipes.

From above equations the laminar head loss as followes

$h_{f(laminar)} = \frac{64}{Re} \frac{L}{d} \frac{V_{av}^2}{2g} = \frac{32 \mu L V_{av}}{\rho g d^2} = \frac{128 \mu L Q}{\pi \rho g d^4}$ (23)

From Eq. 4

$p_1 - p_2 = \frac{4 V_{max} \mu L}{R^2} = \frac{32 V_{av} \mu L}{d^2}$

Pressure drop per unit weight

$h_f = \frac{\Delta p}{\rho g} = \frac{32 \mu L V_{av}}{\rho g\ d^2}$ for laminar flow (24)

Ex.2

An oil of viscosity 0.9 Ns/m^2 and S.G. 0.9 is flowing through a horizontal pipe of 60 mm diameter. If the pressure drop in 100 m length of the pipe is 1800kN/m^2 , determine:

(i) The rate of flow of oil.
(ii) The center-line velocity.
(iii) The total friction drags over 100 m length.
(iv) The power required to maintain the flow.
(v) The velocity gradient at the pipe wall.

(vi) the velocity and shear stress at 8 mm from the wall

Sol.
Area of the pipe,
$A = \frac{\pi}{4} * (0,06)^2 = 2.827 * 10^{-3} (m^2)$ Pressure drop in (100m) length of the pipe, $\Delta p = 1800\ kN/m^2$
i) the rate of flow,Q
$p_1 - p_2 = \Delta p = \frac{32\mu V_{av} L}{d^2}$
$V_{av} = \frac{\Delta p\ d^2}{32\mu L}$
$\therefore\ V_{av} = \frac{1800*10^3*(0.06)^2}{32*0.9*100} = 2.25\ \frac{m}{s}$
Reynolds number, $Re = \frac{\rho V d}{\mu} = \frac{0.9*1000*2.25*0.06}{0.9} = 135$
As Re is less than 2000, the flow is laminar and the rate of flow is,
$Q = A * V_{av} = 2.827 * 10^{-3} * 2.25 = 6.36 * 10^{-3} \frac{m^3}{S} = 6.36 \frac{lit}{s}$
ii) the center-line velocity , V_{max}
$V_{max} = 2V_{av} = 2 * 2.25 = 4.5 \frac{m}{s}$
iii) the total frictional drag over (100m) length
From $\tau_0 = \frac{(p_1 - p_2)d}{4L}$
$\therefore\ \tau_0 = 1800 * 10^3 * \frac{0.06}{4*100} = 270\ N/m^2$
$\therefore$ Friction drag for (100m) length
$F_d = \tau_0 * A_s = \tau_0 * \pi d L = 270 * \pi * 0.06 * 100$
$F_d = 5089\ N$
(iv) The power required to maintain the flow, P,
$P = F_d * V_{av} = 5089 * 2.25 = 11451\ W$
= 15.35 h.p
Alternatively,
$P = Q.\Delta p = 0.00636 * 1800 * 10^3 = 11448\ W$
(v) The velocity gradient at the pipe wall, $\left(\frac{du}{dy}\right)_{y=0}$;
$\tau_0 = \mu.\left(\frac{\partial u}{\partial y}\right)_{y=0}$
$or\ \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\tau_0}{\mu} = \frac{270}{0.9} = 300\ s^{-1}$
(vi) the velocity and shear stress at (8mm) from the wall,
$V = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1 - \frac{r^2}{R^2}\right)$
Or $V = -\frac{1}{4\mu} . \frac{\partial p}{\partial x}\ (R^2 - r^2)$
Here, $y = 8mm = 0.008m$
But y = R-r
$\therefore 0.008 = 0.03 - r ---\rightarrow r = 0.022m$
$\therefore\ V_{(8mm)} = +\frac{1}{4*0.9} * \frac{1800*10^3}{100}\ (0.03^2 - 0.022^2) = 2.08 \frac{m}{s}$

For linear relation $\frac{\tau}{r} = \frac{\tau_0}{R} - - - \rightarrow \ \tau_{(8mm)} = r * \frac{\tau_0}{R} = 0.022 * \frac{270}{0.03} = 198 \ N/m^2$

Or $\tau = \frac{\Delta p}{2L} * r \quad from\ Eq. 7.27$

$\tau = 1800 * 10^3 * \frac{0.022}{2*100} = 198 \ \frac{N}{m^2}$

Table 1: Summary of used equations in pipe

Velocity in circular pipe.	$V_x = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left[1 - \frac{r^2}{R^2}\right]$
V_{max} (max. velocity)	$V_{max} = 2V_{av}$
V_{av} (Average velocity)	$V_{av} = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = \frac{1}{2}V_{max}$
Pressure loss along pipe	$\frac{\Delta p}{L} = 4V_{max}\frac{\mu}{R^2} = \frac{32\mu V_{av}}{d^2}$
Wall shear stress	$\tau_0 = \frac{(p_1 - p_2)d}{4L}$
Shear stress at any ***r***	$\tau = \frac{(p_1 - p_2)r}{2L}$
Energy losses	$h_f = \frac{4\tau_0 L}{\rho g d}$
Energy loss by friction factor	$h_f = f\frac{L}{d}\frac{V^2}{2g}$
Hydraulic diameter	$d_h = \frac{4 \quad Area}{wetted\ primeter}$
Energy loss in Laminar flow	$h_{f\ laminar} = \frac{64}{R_e}\frac{L}{d}\frac{V_{av}^2}{2g} = \frac{32\mu LV}{\gamma d^2} = 128\mu LQ/\pi\rho g d^4$