Lecture-Nine

Introduction to Boundary Layer

1- *Boundary Layer Definitions and Characteristics.*

Boundary Layer is a small region developing around a boundary surface of body, in which the velocity of the flowing fluid increase rapidly from zero at the boundary surface and approaches the velocity of main stream. The layer adjacent to the boundary surface is known as boundary Layer (B.L.). Firstly Introduced by L.Prandtl in 1904. Fluid medium around bodies moving in fluids, can be divided into following two regions

(i) A thin layer adjoining the boundary called B.L where the viscous shear takes place.

(ii) A region outside the boundary layer where the flow behavior is quite like that of an ideal fluid and the potential flow theory is applicable.

U_∞ : is the velocity at the outer edge of the B.L.

δ: is called the dynamic B.L thickness where u=0.99U∞ as shown in Fig.(1).

T_w : is the wall temperature where the fluid immediately at the surface is equal to the temperature of the surface.

δ_T: is the thermal B.L thickness, where the temperatures are changing as $T = T_w \; at \; y = 0$ $T = T_\infty \; at \; y = \delta_T$

In dynamic B.L. u=u(y) u=0 at y=0; U=U_∞ $at \; y = \delta,$

$\delta = \delta(x) \quad and \quad \delta_T = \delta_T(x)$

$\tau_0 = \mu(\frac{\partial u}{\partial y})_w$ is the shear stress at the wall. (1)

The displacement of the streamlines (δ_d) in the free stream as a result of velocity deficits in the B.L is known the displacement thickness. The momentum layer thickness (θ) is the equivalent thickness of a fluid layer with velocity U∞ with momentum equal to the momentum lost due to friction , and is defind as the momentum thickness θ.

$q_w = -k\left(\frac{\partial T}{\partial y}\right)_w$ is the heat flux at the wall. (2)

Both (τ_w, q_w) are function of distance from the leading edge

$\tau_w = \tau_w(x), q_w = q_w(x)$

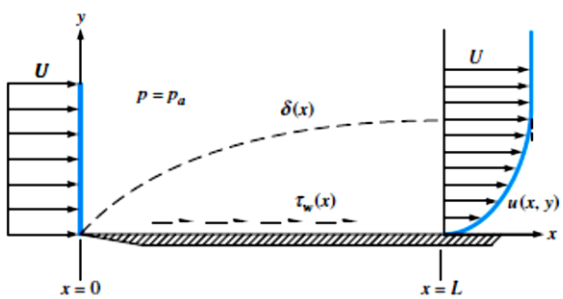

Figure 1: Growth of a boundary layer on a flat plate.

2- Boundary Layer Theory (Flow over Flat Plate).

For flow over a flat plate with zero pressure gradient, the transition process occurs when $Re = \frac{U_\infty x_L}{\nu} = 3 * 10^5$. We assume that $U_\infty = 1.0 \frac{m}{s}$ over a thin plate 10 long $Re_{critical} = 3 * 10^5$

$Re = \frac{U_\infty x_L}{\nu} = \frac{1.0 * x_L}{1.6 * 10^{-5}} = 3 * 10^5$

$x_L = 4.8m$. From Fig.(2), the following cases can be show.

(i) The thickness of B.L. (δ) increases with distance from leading edge

$x, \delta - - \rightarrow \delta(x)$

(ii) δ Decreases as U increases.

(iii) δ Increases as kinematic viscosity υ increases.

(iv) $\tau_0 \approx \mu\left(\frac{U}{\delta}\right); hence\ \tau_0$ decreases as x-increases. However, when B.L becomes turbulent.

v) If $Re = \frac{U_\infty x}{\nu} < 5 * 10^5$ B.L is laminar (Velocity distribution is parabolic)

If $Re = \frac{U_\infty x}{V} > 5 * 10^5$ B.L is turbulent on that portion (Velocity distribution follows log law as a power law).

vi) The critical value of $\frac{Ux}{\nu}$ at which B.L change from laminar to turbulent depends on:

- Turbulence in ambient flow.
- Surface roughness.
- Pressure gradient.
- Plate curvature.
- Temperature difference between fluid and boundary.

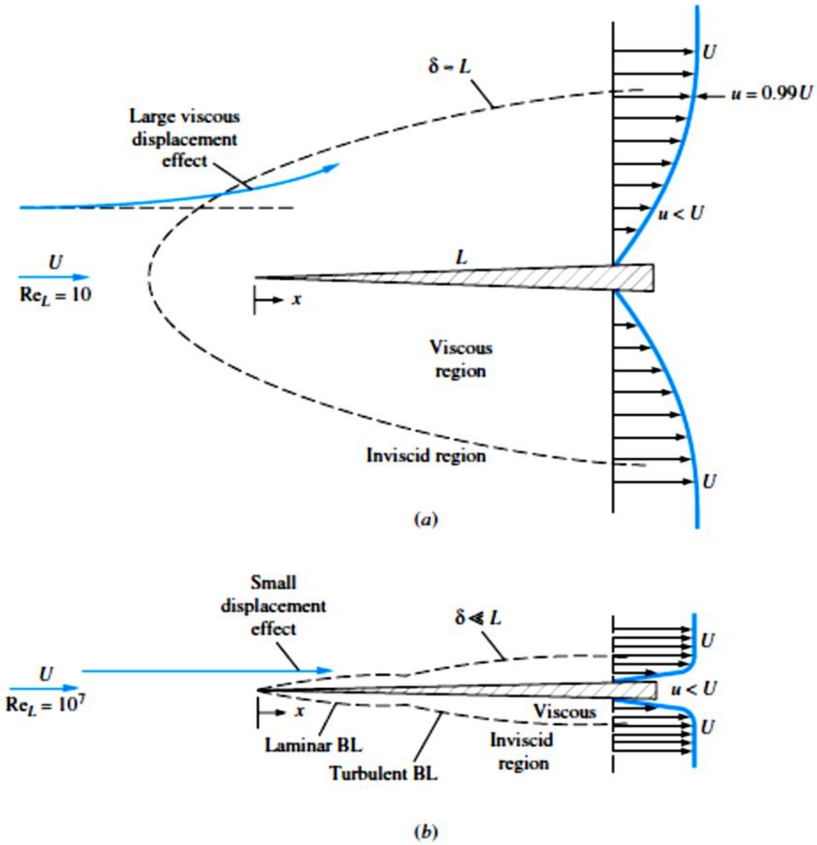

Figure 2: Comparison of flow past a sharp flat plate at low and high Reynolds numbers
(a) Laminar, low –Re flow; (b) high-Re flow.

3- *Displacement Thickness (δ_d) of B.L.*

Consider the mass flow per unit depth across the vertical line y=0 and $y = y_1$ as shown in Fig.(3).

A= actual mass flow between 0 and $y_1 = \int_0^{y_1} \rho u \, dy$

B= theoretical mass flow between 0 and y_1 if B.L were not present

B = $\int_0^{y_1} \rho_\infty U_\infty \, dy$

B-A= decrement in mass flow due to presence of B.L, i.e (missing mass flow)= $\int_0^{y_1} (\rho_\infty U_\infty - \rho u) dy - - - (a)$

Express this missing mass flow as the product of $\rho_\infty U_\infty$ and a heigh (δ_d) that is

Missing mass flow = $\rho_\infty U_\infty \delta_d - - - (b)$

Equating Eq's (a & b)

$$\rho_\infty U_\infty \delta_d = \int_0^{y_1} (\rho_\infty U_\infty - \rho u) dy$$

$$\delta_d = \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right) dy \qquad if\ \rho = \rho_\infty \ \&\ y_1 = \delta$$

$\therefore\ \delta_d = \int_0^{\delta}\left(1-\frac{u}{U_\infty}\right)dy$ (3)

Physically;

1) Missing mass flow
2) Deflected the streamline upward throught a distance δ_d

$\therefore\ \delta_d$ is the distance through which the external inviscid flow is displaced by the presence of the B.L.

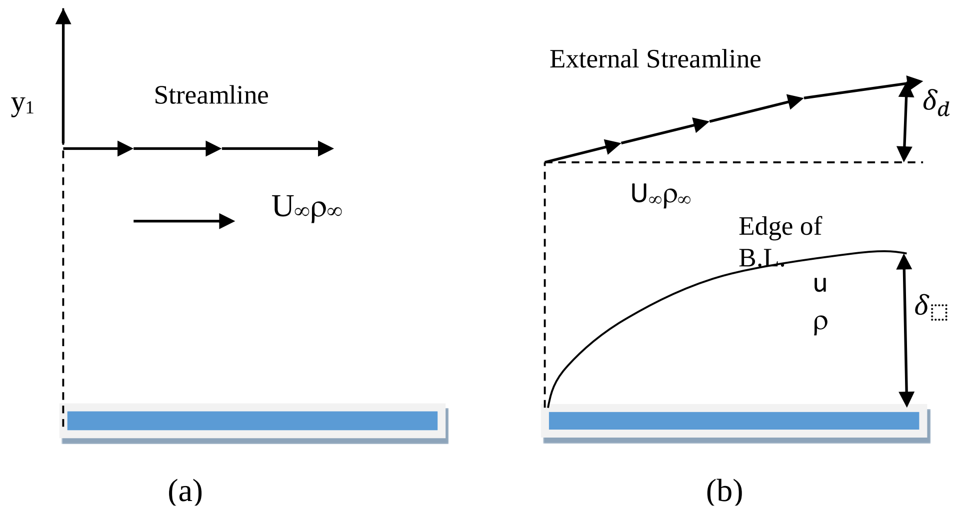

Figure 3: (a) Hypothetical flow with no B.L.(inviscid case).
(b)Displacement thickness in actual flow with B.L.

4- *Momentum Thickness (θ).*

To understand the physical interpretation of θ as the momentum thickness. Consider the mass flow across a segment dy as in Fig. 8.4, given by $dm = \rho u\, dy$ then

A= momentum flow across $dy = dm\, u = \rho u^2 dy ---(a)$

If this same elemental mass flow were associated with the free- stream velocity, where the velocity is U_∞, then

B= momentum flow at free stream velocity associated with ***dm***

$= dmU_\infty = (\rho u\, dy)U_\infty ------(b)$

Hence, B-A= decrement in momentum flow (missing momentum flow) associated with mass ***dm***

$= (\rho u\, U_\infty - \rho u^2)dy = \rho u\,(U_\infty - u)dy ---(c)$

The total decrement in momentum flow across the vertical line from y=0 to $y = y_1$ is the integral of Eq. (c)

$\therefore missing\ momentum\ flow = \int_0^{y_1}\rho u(U_\infty - u)dy ----(d)$

Assume that the missing momentum flow is the product of $\rho_\infty\, U_\infty^2$ and a height θ. Then

$missing\ momentum\ flow = \rho_\infty\, U_\infty^2\theta ----(e)$

Equating Eq's(d& e)

$\rho_\infty\, U_\infty^2\theta = \int_0^{y_1}\rho u(U_\infty - u)dy$

$\therefore\ \theta = \int_0^{y_1}\frac{\rho u}{\rho_\infty U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \qquad if\ \rho = \rho_\infty\ and\ y_1 = \delta$

$\therefore\ \theta = \int_0^\delta \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$ (4)

$\therefore \theta$ is an index that is proportional to the decrement in momentum flow due to the presence of the B.L. It is the height of an ideal stream tube which is carrying the missing momentum flow at free stream conditions.

5- Energy Thickness(δ_e).

Energy thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy K.E. of the flowing fluid.

The mass of flow per second through the elementary strip= $\rho u\, dy$

K.E. of this fluid inside the B.L. $=\frac{1}{2}\dot{m}u^2 = \frac{1}{2}(\rho u\, dy)u^2$

K.E of the same mass of fluid before entering the B.L.

$=\frac{1}{2}(\rho u\, dy)U_\infty^2$; Loss of K.E. through elementary strip is equal to

$=\frac{1}{2}(\rho u\, dy)U_\infty^2 - \frac{1}{2}(\rho u\, dy)u^2$

$=\frac{1}{2}(\rho u)(U_\infty^2 - u^2)dy$

$\therefore Total\ loss\ of\ K.E.of\ fluid = \int_0^\delta \frac{1}{2}\rho u(U_\infty^2 - u^2)dy - - - (i)$

Let δ_e = distance by which the plate is displaced to compensate for the reduction in K.E. then loss of K.E. through $\delta_e$of fluid flowing with velocity U∞ as follows

$=\frac{1}{2}(\rho\, U_\infty \delta_e)U_\infty^2 - - - - - (ii)$

Equating Eq's (i) and (ii), we have

$\frac{1}{2}(\rho U\,\delta_e)U_\infty^2 = \int_0^\delta \frac{1}{2}\rho u\,(U_\infty^2 - u)dy$

$or\ , \delta_e = \frac{1}{U_\infty^3}\int_0^\delta u\,(U^2 - u^2)dy$

$\delta_e = \int_0^\delta \frac{u}{U_\infty}\left(1-\frac{u^2}{U_\infty^2}\right)dy$ (5)

Ex.1

The velocity distribution in the *B.L.* is given by : $\frac{u}{U_\infty} = \frac{y}{\delta}$, where ***u*** is the velocity at a distance ***y*** from the plate and $u = U\ at\ y = \delta, \delta\ being\ B.L.$ thickness. Find

(i)The displacement thickness (ii) the momentum thickness (iii) the energy thickness and (iv) the value of $\delta_d\ /\theta$

Sol.

Velocity distribution $\frac{u}{U_\infty} = \frac{y}{\delta}$

(i) the displacement thickness δ_d

$\delta_d = \int_0^\delta \left(1-\frac{u}{U_\infty}\right)dy$

$= \int_0^\delta \left(1-\frac{y}{\delta}\right)dy$

$=\left[y - \frac{y^2}{2\delta}\right]$

$\delta_d = \left(\delta - \frac{\delta^2}{2\delta}\right) = \delta - \frac{\delta}{2} = \frac{\delta}{2}$

(ii) The momentum thickness, θ:

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

$$= \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy = \left[\frac{y^2}{2y} - \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$\theta = \left[\frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2}\right] = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

(iii) The energy thickness , δ_e:

$$\delta_e = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u^2}{U_\infty^2}\right) dy$$

$$= \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^3}{\delta^3}\right) dy$$

$$\delta_e = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3} = \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}$$

(iv) The value of $\frac{\delta_d}{\theta} = \frac{\frac{\delta}{2}}{\frac{\delta}{6}} = 3$

Ex.2

The velocity distribution in the boundary layer is given by

$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^2}{\delta^2}$ Calculate the following,

(i) The ratio of displacement thickness to B.L thickness $\left(\frac{\delta d}{\delta}\right)$,

(ii) The ratio of momentum thickness to B.L thickness $\left(\frac{\theta}{\delta}\right)$.

Sol.

i) $\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\frac{y^2}{\delta^2}\right) dy$

$$\delta_d = \left[y - \frac{3}{2}\frac{y^2}{2\delta} + \frac{1}{2}\frac{y^3}{3\delta^2}\right]_0^\delta = \left[\delta - \frac{3}{4}\frac{\delta^2}{\delta} + \frac{1}{2} * \frac{\delta^3}{3\delta^2}\right]$$

$$\delta_d = \left(\delta - \frac{3}{4}\delta + \frac{\delta}{6}\right) = \frac{5}{12}\delta \qquad \frac{\delta_d}{\delta} = \frac{5}{12}$$

ii) $\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

$$\theta = \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^2}{\delta^2}\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\frac{y^2}{\delta^2}\right) dy$$

$$\theta = \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{9}{4}\frac{y^2}{\delta^2} + \frac{3}{4}\frac{y^3}{\delta^3} - \frac{1}{2}\frac{y^2}{\delta^2} + \frac{3}{4}\frac{y^3}{\delta^3} - \frac{1}{4}\frac{y^4}{\delta^4}\right) dy$$

$$\theta = \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \left(\frac{9}{4}\frac{y^2}{\delta^2} + \frac{1}{2}\frac{y^2}{\delta^2}\right) + \left(\frac{3}{4}\frac{y^3}{\delta^3} + \frac{3}{4}\cdot\frac{y^3}{\delta^3}\right) - \frac{1}{4}\frac{y^4}{\delta^4}\right] dy$$

$$\theta = \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{11}{4}\frac{y^2}{\delta^2} + \frac{3}{2}\frac{y^3}{\delta^3} - \frac{1}{4}\frac{y^4}{\delta^4}\right] dy$$

$$\theta = \left[\frac{3}{2} * \frac{y^2}{2\delta} - \frac{11}{4} * \frac{y^3}{3\delta^2} + \frac{3}{2} * \frac{y^4}{4\delta^3} - \frac{1}{4} * \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{3}{2} * \frac{\delta^2}{2\delta} - \frac{11}{4} * \frac{\delta^3}{3\delta^2} + \frac{3}{2} * \frac{\delta^4}{4\delta^3} - \frac{1}{4} * \frac{\delta^5}{5\delta^4}\right]$$

$$= \left(\frac{3}{2}\delta - \frac{11}{12}\delta + \frac{3}{8}\delta - \frac{1}{20}\delta\right) = \frac{19}{120}\delta \longrightarrow \frac{\theta}{\delta} = \frac{19}{120}$$