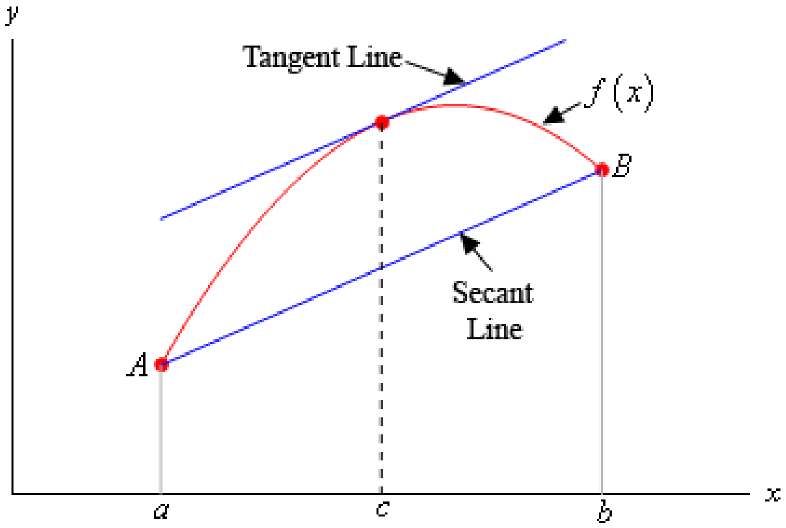

FIGURE 4.22

There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (i.e. the derivative exists) on the interval given.

$$\dot{f}(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the **Mean Value Theorem** all we need to do is plug this into the formula given by the **Mean Value Theorem**.

$$\begin{aligned} \dot{f}(c) &= \frac{f(2) - f(-1)}{2 - (-1)} \\ 3c^2 + 4c - 1 &= \frac{14 - 2}{3} = 4 \\ 3c^2 + 4c - 5 &= 0 \end{aligned}$$

Then,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

Then,

$$c = \frac{-4+\sqrt{76}}{6} = 0.7863 \qquad c = \frac{-4-\sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem.

Then,

$$c = 0.7863$$

Example Suppose that we know that $f(x)$ is continuous and differentiable on $[6, 15]$. Let's also suppose that we know that $f(6) = -2$ and that we know that $\dot{f}(x) \leq 10$. What is the largest possible value for $f(15)$?

Solution

Let's start with the conclusion of the **Mean Value Theorem**.

$$f(15) - f(6) = \dot{f}(c)(15 - 6)$$

Plugging in for the known quantities and rewriting this a little gives,

$$f(15) = f(6) + \dot{f}(c)(15 - 6) = -2 + 9\dot{f}(c)$$

Now we know that $\dot{f}(x) \leq 10$ so in particular we know that $\dot{f}(c) \leq 10$. This gives us the following,

$$\begin{aligned} f(15) &= -2 + 9\dot{f}(c) \\ &\leq -2 + 9(10) \\ &= 88 \end{aligned}$$

All we did was replace $\dot{f}(c)$ with its largest possible value.

This means that the largest possible value for $f(15)$ is 88.

We'll close this section out with a couple of nice facts that can be proved using the **Mean Value Theorem**. Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval $[a, b]$.

Fact 1

If $\dot{f}(x) = 0$ for all x in an interval (a, b) then $f(x)$ is constant on (a, b).

Proof of Fact 1

Take any two $x's$ in the interval (a,b), say x_1 and x_2. Then

$$f(x_2) - f(x_1) = \dot{f}(c)(x_2 - x_1) \tag{4.1}$$

where $x_1 < c < x_2$. But by assumption $\dot{f}(x) = 0$ for all x in an interval (a,b) and so in particular we must have,

$$\dot{f}(c) = 0 \tag{4.2}$$

then,

$$f(x_2) - f(x_1) = 0 \qquad \Rightarrow \qquad f(x_2) = f(x_1) \tag{4.3}$$

Now, since x_1 and x_2 where any two values of x in the interval (a,b) we can see that we must have $f(x_2) = f(x_1)$ for all x_1 and x_2 in the interval and this is exactly what it means for a function to be constant on the interval and so we've proven the fact.

Fact 2
If $\dot{f}(x) = \dot{g}(x)$ for all x in an interval (a,b) then in this interval we have $f(x) = g(x) + c$ where c is some constant.

Proof of Fact 2

If we define,

$$h(x) = f(x) - g(x) \tag{4.4}$$

Then since both $f(x)$ and $g(x)$ are continuous and differentiable in the interval (a,b) then so must be $h(x)$. Therefore the derivative of $h(x)$ is,

$$\dot{h}(x) = \dot{f}(x) - \dot{g}(x) \tag{4.5}$$

However, by assumption $\dot{f}(x) = \dot{g}(x)$ for all x in an interval (a, b) and so we must have that $\dot{h}(x) = 0$ for all x in an interval (a, b). So, by **Fact 1** $h(x)$ must be constant on the interval.

This means that we have,

$$\begin{aligned} h(x) &= c \\ f(x) - g(x) &= c \\ f(x) &= g(x) + c \end{aligned}$$

which is what we were trying to show.

4.8 Optimization

Before proceeding with the examples let's spend a little time discussing some methods for determining if our solution is in fact the absolute minimum/maximum value that we're looking for. In some examples all of these will work while in others one or more won't be all that useful. However, we will always need to use some method for making sure that our answer is in fact that optimal value that we're after.

4.8.1 Method 1

Use the method used in **Finding Absolute Extrema**.

Recall that in order to use this method the range of possible optimal values, let's call it I, must have finite endpoints. Also, the function we're optimizing (once it's down to a single variable) must be continuous on I, including the endpoints. If these conditions are met then we know that the optimal value, either the maximum or minimum depending on the problem, will occur at either the endpoints of the range or at a critical point that is inside the range of possible solutions.

There are two main issues that will often prevent this method from being used however. First, not every problem will actually have a range of possible solutions that have finite endpoints at both ends. We'll see at least one example of this as we work through the examples. Also, many of the functions we'll be optimizing will not be continuous once we reduce them down to a single variable and this will prevent us from using this method.

4.8.2 Method 2

Use a variant of the **First Derivative Test**. So, the definition of this method is,

First Derivative Test for Absolute Extrema

Let I be the interval of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on I , except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that c is in the interval I. If we restrict x to values from I (i.e. we only consider possible optimal values of the function) then,

1. If $\dot{f}(x) > 0$ for all $x < c$ and if $\dot{f}(x) < 0$ for all $x > c$ then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval I.

2. If $\dot{f}(x) < 0$ for all $x < c$ and if $f(x) > 0$ for all $x > c$ then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval I.

4.8.3 Method 3

Use the second derivative.

The summary of this method is: