Lecture Nine

Intensity of Radiation and Lambert's Law

1- Intensity of Radiation (I).

Is defined as the rate of energy leaving a surface in a given direction per unit solid angle per unit area of the emitting surface normal to the mean direction in space.

$$I = \frac{dq}{cos\theta . dA_1 . d\omega}$$

i- **Lambert's Cosine Law:**
A surface is said to obey Lambert's cosine law if the intensity, I, is uniform in all directions. This is an idealization of real surfaces as seen by the emissivity at different zenith angles:

ii- **Solid Angle.** Is defined as a portion of the space inside a spherical enclosed by a conical surface with the vertex of the cone at the center of the sphere. It is measured by the ratio of the spherical surface enclosed by the cone to the square of the radius of the sphere.

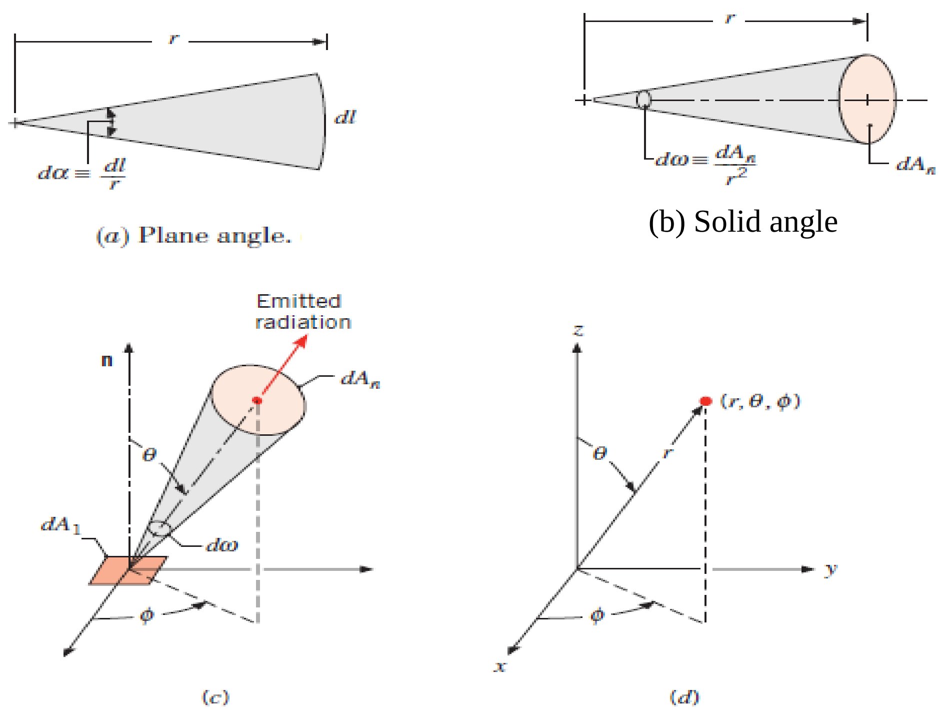


Figure 1 Mathematical definitions. (*a*) Plane angle. (*b*) Solid angle. (*c*) Emission of radiation from a differential area dA_1 into a solid angle $d\omega$ subtended by dA_n at a point on dA_1. (*d*) The spherical coordinate system.

The differential solid angle $d\omega$ is defined by a region between the rays of a sphere and is measured as the ratio of the area dA_n on the sphere to the sphere's radius squared. Accordingly,

$$d\omega \equiv \frac{dA_n}{r^2} \tag{1}$$

The direction may be specified in terms of the zenith and azimuthal angles, θ and φ, respectively, of a spherical coordinate system (Fig.(1.*d*). The area dA_n, through which the radiation passes, subtends a differential solid angle $d\omega$ when viewed from a point on dA_1. As shown in Fig.(2)

the area dA_n is a rectangle of dimension $r\,d\theta \times r\sin\theta\,d\phi$; thus, $dA_n = r^2 \sin\theta\,d\theta\,d\phi$. Accordingly,

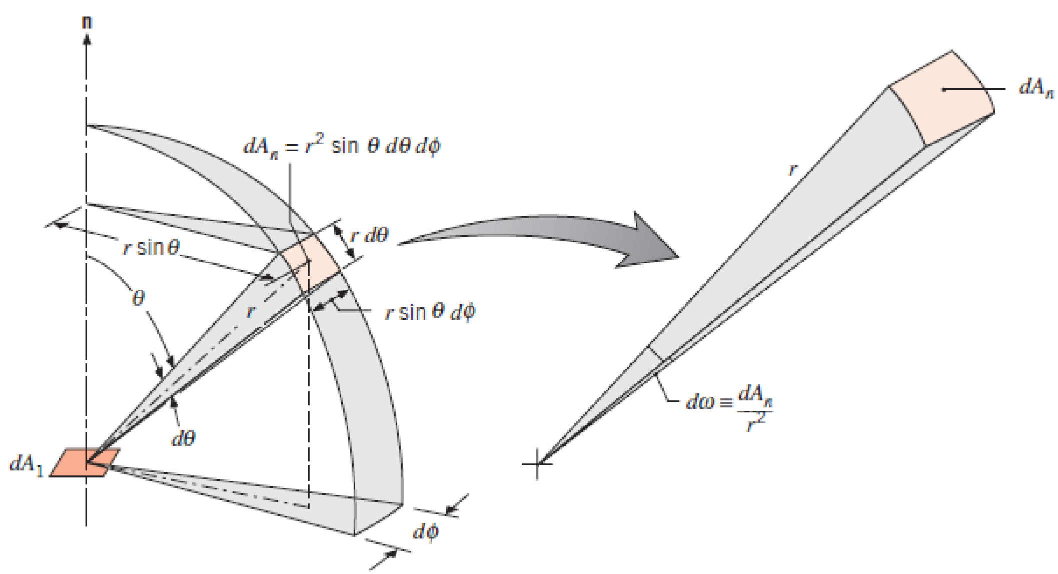


Figure 2: The solid angle subtended by dA_n at a point dA_1 in the spherical

$$d\omega = \sin\theta\, d\theta\, d\phi \tag{2}$$

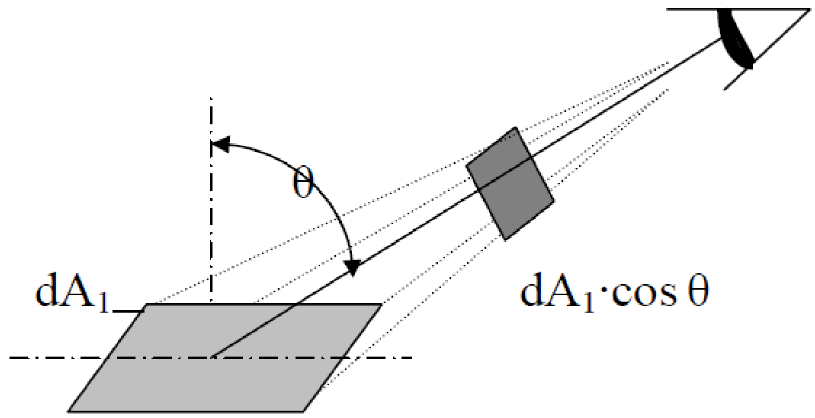
When viewed from a point on an *opaque* surface area element dA_1, radiation may be emitted into any direction defined by a hypothetical hemisphere above the surface. The solid angle associated with the entire hemisphere may be obtained by integrating Eq. (2) over the limits ($\varphi = 0$ to $\varphi = 2\pi$ and $\theta = 0$ to $\theta = \pi/2$). Hence,

$$\omega = \int_h d\omega = \int_0^{2\pi}\int_0^{\pi/2} \sin\theta\, d\theta\, d\phi = 2\pi \int_0^{\pi/2} \sin\theta\, d\theta = 2\pi \text{ sr} \tag{3}$$

iii- **Projected Area.**

The area, dA_1, as seen from the prospective of a viewer, situated at an angle θ from the normal to the surface, will appear somewhat smaller, as $\cos\theta \cdot dA_1$. This smaller area is termed the projected area.

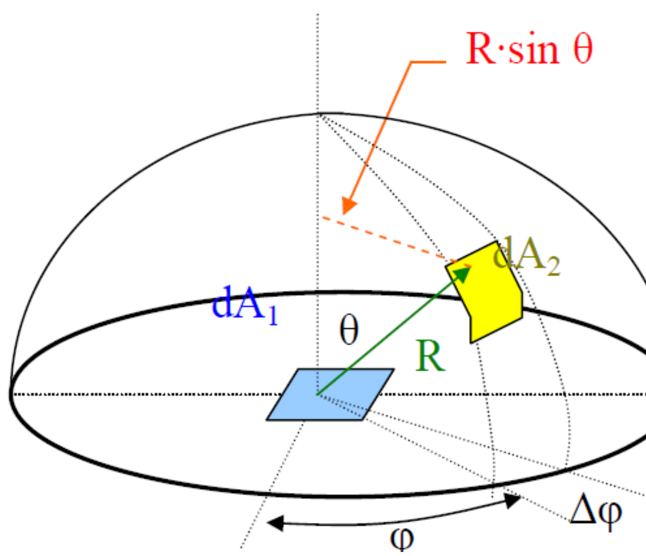
$$A_{projected} = \cos\theta \cdot A_{normal}$$



2- Spherical Geometry.

Since any surface will emit radiation outward in all directions above the surface, the spherical coordinate system provides a convenient tool for analysis. The three basic coordinates shown are R, φ, and θ, representing the radial, azimuthal and zenith directions.

In general dA_1 will correspond to the emitting surface or the source. The surface dA_2 will correspond to the receiving surface or the target. Note that the area proscribed on the hemisphere, dA_2, may be written as



$$dA_2 = [(R \cdot \sin\theta) \cdot d\varphi] \cdot [R \cdot d\theta]$$

or, more simply as:

$$dA_2 = R^2 \cdot \sin\theta \cdot d\varphi \cdot d\theta]$$

Recalling the definition of the solid angle,

3- Relationship Between Emissive Power and Intensity.

By definition of the two terms, emissive power for an ideal surface, E_b, and intensity for an ideal surface, I_b.

$$E_b = \int_{hemisphere} I_b . cos\theta . d\omega$$

Replacing the solid angle by its equivalent in spherical angles:

$$E_b = \int_0^{2\pi} \int_0^{\pi/2} I_b . cos\theta . sin\theta . d\theta d\varphi$$

Integrate once, holding I_b constant:

$$E_b = 2 \cdot \pi \cdot I_b \cdot \int_0^{\pi/2} \cos\theta \cdot \sin\theta \cdot d\theta$$

Integrate a second time. (Note that the derivative of sin θ is cos θ·dθ.)

$$E_b = 2 \cdot \pi \cdot I_b \cdot \left. \frac{\sin^2\theta}{2} \right|_0^{\pi/2} = \pi \cdot I_b$$

$$\boxed{E_b = \pi \cdot I_b}$$

4- Radiation Exchange.

During the previous section we introduced the intensity, I, to describe radiation within a particular solid angle.

$$I = \frac{dq}{\cos\theta \cdot dA_1 \cdot d\Omega}$$

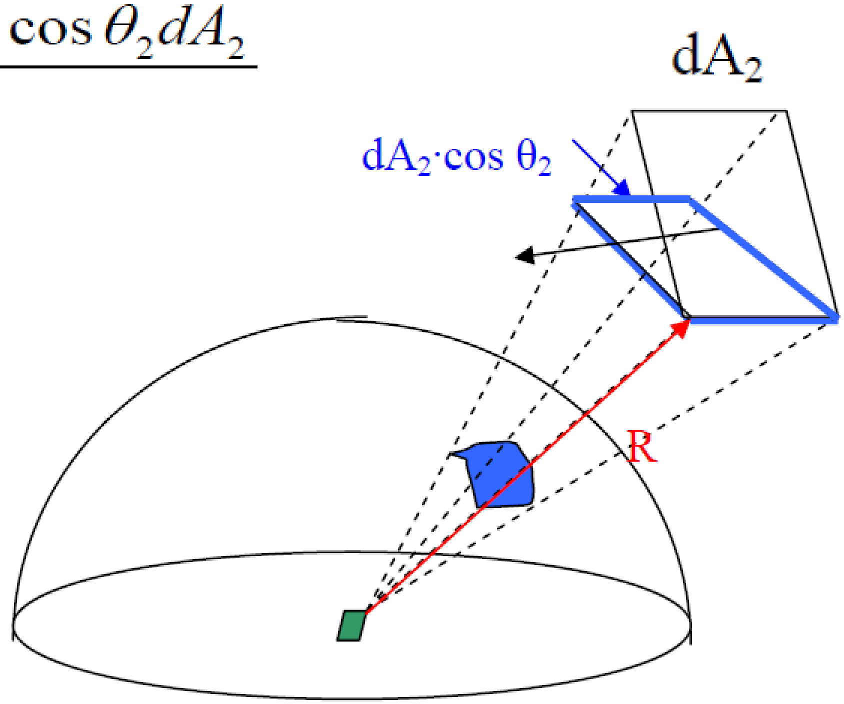
This will now be used to determine the fraction of radiation leaving a given surface and striking a second surface.

Rearranging the above equation to express the heat radiated:

$$dq = I \cdot \cos\theta \cdot dA_1 \cdot d\Omega$$

Next we will project the receiving surface onto the hemisphere surrounding the source. First find the projected area of surface dA_2, $dA_2 \cdot \cos\theta_2$. (θ_2 is the angle between the normal to surface 2 and the position vector, R.) Then find the solid angle, (ω), which encompasses this area. Substituting into the heat flow equation above:

$$dq = \frac{I \cdot \cos\theta_1 \cdot dA_1 \cdot \cos\theta_2 dA_2}{R^2}$$



To obtain the entire heat transferred from a finite area, dA_1, to a finite area, dA_2, we integrate over both surfaces:

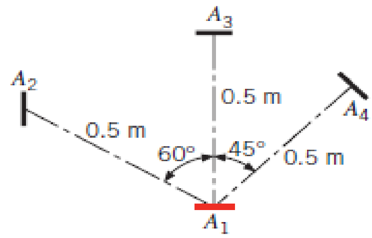
$$q_{1\to 2} = \int_{A_2}\int_{A_1} \frac{I \cdot \cos\theta_1 \cdot dA_1 \cdot \cos\theta_2 dA_2}{R^2}$$

To express the total energy emitted from surface 1, we recall the relation between emissive power, E, and intensity, I.

$$q_{emitted} = E_1 \cdot A_1 = \pi \cdot I_1 \cdot A_1$$

EXAMPLE 1

A small surface of area $A_1 = 10^{-3}$ m^2 is known to emit diffusely, and from measurements the total intensity associated with emission in the normal direction is $I_n = 7000$ W/m^2. sr.



Radiation emitted from the surface is intercepted by three other surfaces of area $A_2 = A_3 = A_4 = 10^{-3}$ m^2, which are 0.5 m from A_1 and are oriented as shown. What is the intensity associated with emission in each of the three directions? What are the solid angles subtended by the three surfaces when viewed from A_1? What is the rate at which radiation emitted by A_1 is intercepted by the three surfaces?

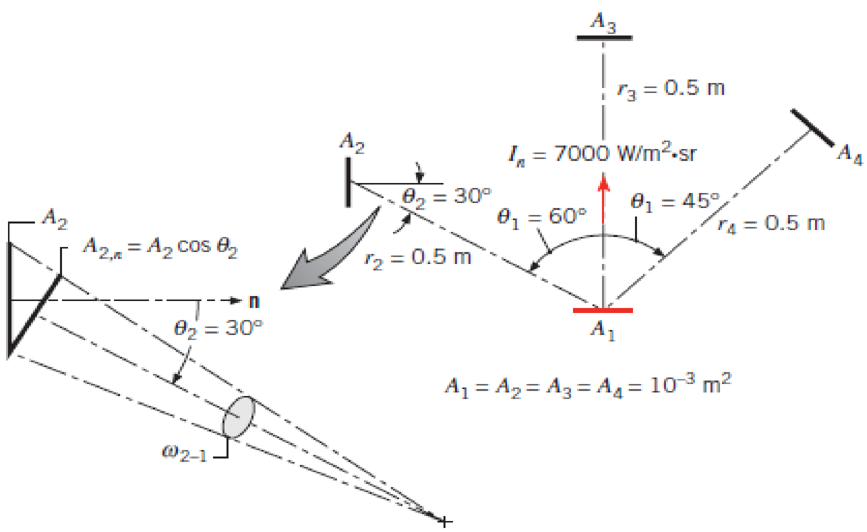
Solution

Known: Normal intensity of diffuse emitter of area A_1 and orientation of three surfaces relative to A_1.

Find:

1. Intensity of emission in each of the three directions.
2. Solid angles subtended by the three surfaces.
3. Rate at which radiation is intercepted by the three surfaces.

Schematic:



Assumptions:

1. Surface A_1 emits diffusely.
2. A_1, A_2, A_3, and A_4 may be approximated as differential surfaces, $(A_j/r_j^2) \ll 1$.

Analysis:

1. From the definition of a diffuse emitter, we know that the intensity of the emitted radiation is independent of direction. Hence

$$I = 7000\,\text{W/m}^2 \cdot \text{sr} \qquad \triangleleft$$

for each of the three directions.

2. Treating A_2, A_3, and A_4 as differential surface areas, the solid angles may be computed from Equation 12.7

$$d\omega \equiv \frac{dA_n}{r^2}$$

where dA_n is the projection of the surface normal to the direction of the radiation. Since surfaces A_3 and A_4 are normal to the direction of radiation, the solid angles subtended by these surfaces can be directly found from this equation as

$$\omega_{3-1} = \omega_{4-1} = \frac{A_3}{r^2} = \frac{10^{-3}\ \text{m}^2}{(0.5\ \text{m})^2} = 4.00 \times 10^{-3}\ \text{sr} \qquad \triangleleft$$

Since surface A_2 is not normal to the direction of radiation, we use $dA_{n,2} = dA_2 \cos\theta_2$, where θ_2 is the angle between the surface normal and the direction of the radiation. Thus

$$\omega_{2-1} = \frac{A_2 \cos\theta_2}{r^2} = \frac{10^{-3}\ \text{m}^2 \times \cos 30^\circ}{(0.5\,\text{m})^2} = 3.46 \times 10^{-3}\ \text{sr} \qquad \triangleleft$$

3. Approximating A_1 as a differential surface, the rate at which radiation is intercepted by each of the three surfaces may be found from Equation 12.11, which, for the total radiation, may be expressed as

$$q_{1-j} = I \times A_1 \cos\theta_1 \times \omega_{j-1}$$

where θ_1 is the angle between the normal to surface 1 and the direction of the radiation. Hence

$$\begin{aligned} q_{1-2} &= 7000\ \text{W/m}^2 \cdot \text{sr}\ (10^{-3}\ \text{m}^2 \times \cos 60^\circ)\ 3.46 \times 10^{-3}\ \text{sr} \\ &= 12.1 \times 10^{-3}\ \text{W} \qquad \triangleleft \end{aligned}$$

$$\begin{aligned} q_{1-3} &= 7000\ \text{W/m}^2 \cdot \text{sr}\ (10^{-3}\ \text{m}^2 \times \cos 0^\circ)\ 4.00 \times 10^{-3}\ \text{sr} \\ &= 28.0 \times 10^{-3}\ \text{W} \qquad \triangleleft \end{aligned}$$

$$\begin{aligned} q_{1-4} &= 7000\ \text{W/m}^2 \cdot \text{sr}\ (10^{-3}\ \text{m}^2 \times \cos 45^\circ)\ 4.00 \times 10^{-3}\ \text{sr} \\ &= 19.8 \times 10^{-3}\,\text{W} \qquad \triangleleft \end{aligned}$$