LIMIT OF FUNCTIONS AND CONTINUITY

Definition (Limit)-Let (X, d_1) and (Y, d_2) be two metric spaces, suppose that $S \subset X$, $f: S \longrightarrow Y$, we say that $f(x)$ **tends** to the limit L as x tends to a if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that

$$d_2(f(x), L) < \epsilon \;\; if \;\; d_1(x, a) < \delta$$

and we write **:** $\mathbf{lim}_{x \to a} \boldsymbol{f(x) = L}$

Example: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, defined as $f(x) = 3x - 1$, prove that $\mathbf{lim}_{x \to 2} \boldsymbol{f(x) = 5}$

Solution: L=5, a=2 $d_1(x, y) = d_2(x, y) = |x - y|$

$\forall\, \epsilon > 0$, to find $\delta > 0$,such that $d_2(f(x), 5) < \epsilon$ whenever $d_1(x, 2) < \delta$, if $d_1(x, 2) = |x - 2| < \delta$

$$d_2(f(x), 5) = |f(x) - 5| = |3x - 1 - 5| = |3x - 6| = 3|x - 2| < 3\delta$$

Choose $\delta = \frac{\epsilon}{3}$, to get : $\lim_{x \to 2} f(x) = 5$

Theorem (1)(without proof) : ***If f has a limit L then it is unique.***

Example: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, Such that:

1- $f(x) = x^2$, prove that $\lim_{x \to a} f(x) = a^2$

2- $f(x) = \sqrt{x}$, prove that $\lim_{x \to a} f(x) = \sqrt{a}$

3- $f(x) = \frac{1}{1+x}$, prove that $\lim_{x \to 1} f(x) = \frac{1}{2}$

4- $f(x) = \sin x$, prove that $\lim_{x \to a} f(x) = \sin a$

Properties of limit of functions

Let $S \subset \mathbb{R}^n, f: S \longrightarrow \mathbb{R}^m, and\ g: S \longrightarrow \mathbb{R}^m$, and $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then:

1- $\lim_{x\to a}(f+g)(x) = A + B$
2- $\lim_{x\to a}(fg)(x) = AB$
3- $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{A}{B}, \quad g(x) \neq 0 \quad and\ B \neq 0$

Definition (Continuity)-Let (X, d_1) and (Y, d_2) be two metric spaces, suppose that $S \subset X, p \in X\ and\ f: S \longrightarrow Y$, we say that $f(x)$ is **continuous** at $x = p$, if $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that

$$d_2(f(x), f(p)) < \epsilon \ \ whenever\ \ d_1(x, p) < \delta$$

Remark: If $f(x)$ is **continuous** at x , if $\forall\ x \in S$, then $f(x)$ is **continuous** on S

Notation: By definition , if $f(x)$ is **continuous** at p, then:

1- $\forall\ x \in S\ \ d_1(x,p) < \delta \to x \in B(p,\delta) \to f(x) \in f\big(B(p,\delta)\big)$
2- $\forall \epsilon > 0\ d_2\big(f(x), f(p)\big) < \epsilon \to f(x) \in B(f(p), \epsilon) \to f(B(p,\delta) \subseteq f(B(f(p), \epsilon))$

Example: Prove that $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$, is continuous $\forall\ x$, with $d_1(x,y) = d_2(x,y) = |x - y|$

Solution : Let $p \in \mathbb{R}$ to prove that f is continuous at $x = p$, let $\epsilon > 0, to\ find\ \delta > 0$ such that $|f(x) - f(p)| < \epsilon$ if $|x - p| < \delta$.

$$|f(x) - f(p)| = |x^2 - p^2| = |(x-p)(x+p)| = |(x-p)||(x+p)| < |(x-p)|(|x| + |p|) \dots (1)$$

Since $\big||x| - |p|\big| \leq |x - p| < \delta \Rightarrow \big||x| - |p|\big| < \delta \Rightarrow -\delta < |x| - |p| < \delta \Rightarrow |x| < \delta + |p|$

Substitute in (1) to get:

$$|f(x) - f(p)| \leq |x - p|(|x| + |p|) < |x - p|(|p| + |p| + \delta)$$

Choose $\delta = \min\left\{1, \frac{\epsilon}{2|p|+1}\right\}$

$$\Rightarrow |f(x) - f(p)| < \delta(2|p| + 1) = \frac{\epsilon}{2|p| + 1} \cdot 2|p| + 1 = \epsilon$$

Then f is continuous at $x = p, \forall\, p \in \mathbb{R}$,

$\Rightarrow f$ is continuous on $\mathbb{R}$.

H.W 1: If $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$, prove that f is continuous at $x = 3$

H.W 2: If $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$, prove that f is continuous at $x = 0$

Theorem (2) ***Let (X, d_1) and (Y, d_2) be two metric spaces, and $f: X \longrightarrow Y$, then $f(x)$ is continuous at $p \in X$, iff $\lim_{n\to\infty} f(x_n) = f(p)$ for each sequence $\langle x_n \rangle$ in X and $x_n \to p$***

Proof: $\Rightarrow$ **if** $f(x)$ is continuous at $p \in X$, and $\langle x_n \rangle$ is a sequence in X such that $x_n \to p$. To prove that $\lim_{n\to\infty} f(x_n) = f(p)$

(i.e. $\exists\, k \in \mathbb{N} \ni d_2(f(x_n), f(p)) < \epsilon, \forall n > k$)

Since f is continuous at p, then $\forall \epsilon > 0$, $\exists\, \delta > 0 \ni d_2(f(x), f(p)) < \epsilon,\ if\ \ d_1(x,p) < \delta\ , \forall x$.

Since $x_n \to p$, then $\exists\, k \in \mathbb{N} \ni d_1(x_n, p) < \epsilon$, $\forall n > k$, since f is continuous , then at this k we have $d_2\big(f(x_n), f(p)\big) < \epsilon, \forall n > k$

Then $\lim_{n\to\infty} f(x_n) = f(p)$

Conversely, $\Longleftarrow$, suppose that $f(x_n) \to f(p)$ for each $x_n \to p$, to prove that f is continuous at p.

Suppose that f is not continuous at p , then:

$\exists \epsilon > 0, \exists x \in X$ such that $d_2\big(f(x), f(p)\big) \geq \epsilon$ and $d_1(x,p) < \delta$

Choose $\delta_1 = 1 \Longrightarrow \exists x_1 \in X \ni d_2\big(f(x_1), f(p)\big) \geq \epsilon$
and $d_1(x_1, p) < \delta_1 = 1$

$\delta_2 = \frac{1}{2} \Longrightarrow \exists x_2 \in X \ni d_2\big(f(x_2), f(p)\big) \geq \epsilon$ and $d_1(x_2, p) < \delta_2 = \frac{1}{2}$

$\delta_3 = \frac{1}{3} \Longrightarrow \exists x_3 \in X \ni d_2\big(f(x_3), f(p)\big) \geq \epsilon$ and $d_1(x_3, p) < \delta_3 = \frac{1}{3}$

$\vdots$

$\delta_n = \frac{1}{n} \Longrightarrow \exists x_n \in X \ni d_2\big(f(x_n), f(p)\big) \geq \epsilon$and $d_1(x_n, p) < \delta_n = \frac{1}{n}$

Since $\frac{1}{n} \to 0 \ as\, n \to \infty \implies \quad d_1(x_n, p) < \frac{1}{n} \to 0$

$\implies \langle x_n \rangle$ converges to p

But $d_2\big(f(x_n), f(p)\big) \geq \epsilon \implies \langle f(x_n) \rangle \nrightarrow f(p) \quad \implies \quad C!$

Then f is continuous at p.

Examples

1- $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$

Use theorem (2) to prove that f is not continuous at $p=0$

Let $\langle x_n \rangle = \langle \frac{1}{n} \rangle \Longrightarrow \frac{1}{n} \to 0\ as\ n \to \infty$

$\forall n, \frac{1}{n} > 0 \ \Longrightarrow f(x_n) = f(\frac{1}{n}) = 1 \neq f(0) = 0$

by theorem 2, f is not continuous at $p = 0$

2- Let $f: [a, b] \longrightarrow \mathbb{R}, f(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ 2 & if\ x \in \mathbb{Q}' \end{cases}$

Use theorem 2 to prove that f is not continuous everywhere.

Let $p \in [a, b] \ \Rightarrow p$ is rational or irrational.

I) If $p \in \mathbb{Q}$,

Since (between any two reals there are infinitely many rationals and irrationals) and(for any real number p there is irrational or rational Cauchy sequence converges to p) then, there is an irrational sequence $\langle x_n \rangle \to p$, Then, $\langle f(x_n)\rangle \to 2$ and $2 \neq f(p) = 1$

Then by theorem (2) f is not continuous at $p \quad \forall\, p \in \mathbb{Q}$

II) Similarly, if $p \in \mathbb{Q}'$, we can show that f is not cont. at $p \quad \forall\, p \in \mathbb{Q}'$

Theorem (3) ***Let*** (X, d_1) ***and*** (Y, d_2) ***be two metric spaces, and*** $f: X \longrightarrow Y$***, then*** $f(x)$ ***is continuous on X, iff*** $f^{-1}(V)$ ***is open set in X for every open set V in Y.***

Proof) $\Longrightarrow$ Suppose that f is cont. on X, let V be an open set in Y, to show that $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V) \Longrightarrow f(x) \in V$

Since V is open set $\Longrightarrow \exists \epsilon > 0 \ni B(f(x), \epsilon) \subset V$

Since f is cont. $\Longrightarrow \exists \delta > 0 \ni f(B(x, \delta)) \subset B(f(x), \epsilon) \subset V$

$$\Longrightarrow f^{-1}\Big(f\big(B(x,\delta)\big)\Big) \subset f^{-1}(V) \Longrightarrow B(x,\delta) \subset f^{-1}(V)$$

This is true for all x in $f^{-1}(V) \Longrightarrow f^{-1}(V)$ is open.

$\Longleftarrow$: Suppose that $f^{-1}(V)$ is open set in Y for each open set V in Y, to prove that f is cont.

Let $\epsilon > 0,\ x \in X$

Let $B(f(x), \epsilon)$ open in Y $\Longrightarrow f^{-1}(B(f(x), \epsilon)\ )$ open in X.

By definition of open set $\Longrightarrow f^{-1}\big(B(f(x), \epsilon)\big), \exists \delta > 0$

$\ni B(f(x), \delta) \subset f^{-1}\big(B(f(x), \epsilon)\big)$

$\Longrightarrow f(B(x, \delta) \subset B(f(x), \epsilon)$

Then by remark, f is cont. at x and this is true for all $x \in X$.

Example

1) Let $f: X \to X, f(x) = x^2$, prove that f is cont. in $\mathbb{R}$
   Let V be open set in Y
   $V = (a, b)$ there are three cases:

Case 1: if $a, b > 0$

$f^{-1}(V) = (\sqrt{a}, \sqrt{b}) \cup (-\sqrt{b}, -\sqrt{a})$ which is union of two open intervals, then $f^{-1}(V)$ is open in X

Case 2: if $a < 0 \ and \ b > 0$

$f^{-1}(V) = (-\sqrt{b}, \sqrt{b})$ then $f^{-1}(V)$ is open in X

Case 3: if $a, b < 0 \Longrightarrow f^{-1}(V) = \Phi \Longrightarrow f^{-1}(V)$ is open in X.

According to above three cases , V is open in Y implies $f^{-1}(V)$ is open in X

Then , by theorem (3) f is cont. on X

2) $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$
3) $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x + 1$
4) $f: \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$, not cont. at $x = 0$

Theorem (5) ***Let X be a metric space and f, g be two real valued functions. If f, g are cont. then:***

1- $f \mp g$ 2- $f \cdot g$ 3-$r f$, $r \in \mathbb{R}$ 4- $\frac{f}{g}$ $g \neq 0$ are cont. functions.

Corollary Every polynomial of the form:

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x \quad + a_n$$

Where a_i are constants $i = 0,1,2, \cdots, n$, $n \in \mathbb{N}$, is cont.

Proof: since $f(x) = x^n$ is cont.$\forall n$ and rf is cont. (by th.4)

$\Longrightarrow$ the sum of cont. functions is cont. (by th.4)

Definition(Vector space)(*review)*

$(V, +, \cdot)$ is a vector space over $\mathbb{R}$ if:

1- $V \neq \Phi$
2- $(V, +)$ is comm. Group.
3- $\cdot$ is scalar product: if $r, s \in \mathbb{R}$ $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$
   a- $(r+s)\boldsymbol{v} = r\boldsymbol{v} + s\boldsymbol{v}$
   b- $r(\boldsymbol{v} + \boldsymbol{w}) = r\boldsymbol{v} + r\boldsymbol{w}$
   c- $(rs)\boldsymbol{v} = r(s\boldsymbol{v})$
   d- $1 \cdot \boldsymbol{v} = \boldsymbol{v}$

Theorem (5) ***Let X be a metric space , define the following set:***

$$\boldsymbol{C}(\boldsymbol{X}) = \{\boldsymbol{f}: \boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}, \qquad \boldsymbol{f} \text{ is continuous}\}$$

Then $\boldsymbol{C}(\boldsymbol{X})$ is a vector space over $\mathbb{R}$.

Theorem (6) ***Let X, Y be two metric spaces, and $f: X \longrightarrow Y$ is continuous function if X is compact then $f(X)$ is compact.***

Proof $f(X) = \{f(x): x \in X\}$

Suppose that $\{V_\lambda : \lambda \in \Lambda\}$ is open cover for $f(X) \Longrightarrow f(X) \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$

To find finite sub-cover for $f(X)$.

Since V_λ is open set in Y and f is cont.

$\Longrightarrow f^{-1}(V_\lambda)$ open in X (by th.3)

Since $f(X) \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$ $X \subseteq f^{-1}(\bigcup_{\lambda \in \Lambda} V_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda)$

$\Longrightarrow \{f^{-1}(V_\lambda), \lambda \in \Lambda\}$ is open cover for X

Since X is compact then there is a finite subcover , say, $\{f^{-1}(V_1), f^{-1}(V_2), \cdots, f^{-1}(V_n)\}$

$$\Longrightarrow X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i) = f^{-1}(\bigcup_{i=1}^{n} V_i)$$

$$\Longrightarrow f(X) \subseteq (\bigcup_{i=1}^{n} V_i)$$

$\Longrightarrow \{V_1, V_2, \cdots, V_n\}$ is finite subcover for $f(X) \rightarrow f(X)$ is compact.

Definition Let X be a metric space and $f: X \rightarrow \mathbb{R}$, we say that f is bounded if $\exists M > 0$, such that $|f(x)| \leq M, \ \forall x \in X$

On the other hand $f(X) = \{f(x): x \in X\}$ is bounded set if it has upper and lower (i.e. f is bounded $\Leftrightarrow f(X)$bounded set)

Theorem (7): ***If f is a cont. mapping of a compact metric space X into $\mathbb{R}$ then $f(X)$ is closed and bounded. Thus, f is bounded.***

Proof :Since f is cont and X is compact by theorem (6) then $f(X)$ is compact then $f(X)$ is closed and bounded Thus, f is bounded

Example: Give an example for bounded function its domain is not compact.

Consider the function $X = (0,1)$ and $f: X \rightarrow \mathbb{R}$ defined as:

$$f(x) = 3x \quad or \ \ f(x) = x^2$$

Note that $|f((0,1))|$ is bounded with $M = 3$ but $\mathbb{R}$ is not compact.

Example: $X = (0, \infty)$ not compact and $f(x) = \frac{1}{x}$ is continuous

But f is not bounded

(since $\forall M > 0, \exists k \in \mathbb{N} \ni M < \frac{1}{k} = f(k)$(by arch. prop.))

Definition (Maximum) Let X be a metric space and $f: X \to \mathbb{R}$, suppose that f is bounded a point a is called:

1- Maximum extreme point of f if $f(x) \leq f(a), \ \forall x \in X$

2- Minimum extreme point of f if $f(a) \leq f(x), \ \forall x \in X$

Example $f: \mathbb{R} \to \mathbb{R}, \ f(x) = \sin(x)$, Find max and min extreme point

f is bounded function, since $-1 \leq \sin(x) \leq 1, \ \forall \ x$

then points: $a = \frac{\pi}{2} \mp 2n\pi \, , n \in \mathbb{Z} \; are \; all \; max \; extreme \; points \; for \; f$

while $a = -\frac{\pi}{2} \mp 2n\pi \, , n \in \mathbb{Z}$ are all min extreme points for f

Example Give an example for a function f which has unique max and unique min.

Theorem (8) ***Let $f: X \to \mathbb{R}$ be continuous function. If X is compact then $\exists a, b \in X$ such that $f(a) \leq f(x) \leq f(b), \forall x \in X$***

(i.e. f has max at b and min at a)

Proof: To prove that $\exists b \in X$ such that $f(x) \leq f(b), \forall x \in X$,

Since f is cont. function on a compact set X then $f(X)$ is compact

$\Longrightarrow f(X)$ is closed and bounded.

Since $f(X)$ is bounded above by completeness axiom $\Longrightarrow f(X)$ has supremum, say $sup(f(X)) = M$

To prove that M is acc. Point of $f(X)$, to prove that $\forall \epsilon > 0$

$(M - \epsilon, M + \epsilon) \backslash \{M\} \cap f(X) \neq \Phi$ if not $\Longrightarrow \exists \epsilon > 0$

Such that $(M - \epsilon, M + \epsilon) \backslash \{M\} \cap f(X) \neq \Phi \Longrightarrow M - \epsilon$ is upper bound of $f(X) \Longrightarrow C!$

Since $M - \epsilon < M = \sup\big(f(X)\big) \Longrightarrow M \; is \; acc.pt$ of $f(X)$

Since $f(X)$ is closed $\to M \in f(X)$ $\quad (S' \subset S \;\Leftrightarrow S \; closed)$

$\Rightarrow \exists b \in X \ni f(b) = M$ and $f(x) \leq M = f(b)$

$\Rightarrow f$ has a max extreme point.

By similar way prove that $\exists a \in X \ni f(a) \leq f(x)$

Definition (Uniform continuity)

Let X be a metric space and $f: X \to \mathbb{R}$, suppose that f is called uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0$, such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta, \forall x, \ \forall y \ \in X$.

Remark the choose of δ in the definition of uniform continuity is depending on ϵ only.

Theorem (9)*Every uniformly continuous function is continuous but the converse is not true.*

Proof: Uniformly cont. $\Rightarrow$ continuous ?

Let f be a uniformly cont. function on X

$\Rightarrow \forall \epsilon > 0, \exists \delta > 0 \ni |f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta, \ \forall x, y$

Take $y = p$

$\Rightarrow \forall \epsilon > 0, \exists \delta > 0 \ni |f(x) - f(y)| < \epsilon$

whenever $d(x, p) < \delta, \forall x \in X \Rightarrow f$ is continuous at p $\forall p$.

Example: To show that cont. $\nRightarrow$ uniformly cont.

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is cont. function

Let $x = n, y = n + \frac{1}{n}, n \in \mathbb{N}$

$d(x, y) = |x - y| = \left|n - (n + \frac{1}{n})\right| = \frac{1}{n} < \delta$

(by arch. For any real $\delta, \exists n \in \mathbb{N} \ni \frac{1}{n} < \delta$)

Take $\epsilon = 1$

$$|f(x)-f(y)| = \left|n^2 - \left(n+\frac{1}{n}\right)^2\right| = \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right|$$

$= 2 + \frac{1}{n^2} > \epsilon = 1 \Longrightarrow f$ is not uniformly cont.

Theorem(10) ***Let $f: X \to \mathbb{R}$, be a continuous function. If X is compact then f is uniformly cont.*** *(without proof)*

Theorem(11)(Intermediate value property (IVP))

Let f be a continuous function on $[a,b]$ and $f(a) = \alpha$, $f(b) = \beta$, then for all γ, $\alpha < \gamma < \beta$ $\exists c$, $a < c < b$ and $f(c) = \gamma$

Proof: Suppose that $S = \{x: x \in [a,b], f(x) \leq \gamma\}$

$S \neq \Phi$ since $a \in S$

S is bounded above since b is an upper bound of S

By completeness axiom ,

S has a supremum, say , $c = \sup(S) \Longrightarrow a < c < b$

Then there are three cases: $f(c) < \gamma$ or $f(c) > \gamma$ or $f(c) = \gamma$

If $f(c) < \gamma$, since f is continuous at c and $f(c) < \gamma \Longrightarrow \exists > 0 \ni f(x) < \gamma \ \forall x \in (c-\epsilon, c+\epsilon) \cap [a,b]$

For this x, if $c < x < b \Longrightarrow x \in S$ and $c < x \Longrightarrow C!$ For $f(c) \nless \gamma$

By similar way $f(c) \ngtr \gamma \Longrightarrow f(c) = \gamma$

Then $\exists c, a < c < b \ni f(c) = \gamma$

Applications of IVP

Theorem (12)(Interval theorem)

If f is cont. in the interval $I = [a,b]$ then $f(I)$ is closed and bounded.

Proof: Since I is closed and bounded then I is compact

Since f is cont. on I then f has max and min extreme points

Then $\exists c, d$ $such\ that$ $f(c) = m$ and $f(d) = M$

Such that $m \leq f(x) \leq M, \forall x \in I$

There are two cases $c < d$ or $d < c$

If $c < d$ apply IVP on f and $[c, d]$

$$\forall y, y \in (m, M), \exists x \in (c, d) \ni f(x) = y \implies f(I) = [m, M]$$

Theorem (13) Fixed point theorem

Let $f: [0,1] \to [0,1]$ be cont. function. Then there is at least number c $\ni f(c) = c$. (c is called fixed point)

Proof Suppose that $g: [0,1] \to \mathbb{R} \ni g(x) = f(x) - x$

g is cont. in $[0,1]$ since f is cont. and $i: X \to X$ is also cont. and the sum of tow cont. functions is cont.

If $f(0) = 0$ anf $f(1) = 1$ this complete the proof.

If $f(0) \neq 0$ and $f(1) \neq 1$ then:

Since f is onto function ($f: [0,1] \to [0,1]$)

Then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$

$\implies g(1) < 0 < g(0)$

Then by IVP we get: $\exists c, 0 < c < 1 \ni g(c) = 0$

$\implies f(c) - c = 0 \implies f(c) = c$

Then f has a fixed point.